# A linear solution for flow over mountains and its comparison with the COSMO model 

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## 1 Introduction

Comparing against analytical solutions for the equations of motion is one of the main testing tools during the development of dynamical cores. The broadest class of analytical solutions are linearisations and the most important ones are the solutions for flow over mountains. Such solutions are well known since several decades (e.g. Queney, 1948, Smith, 1979, 1980). The problem in the application of model testing lies in the choice of the approximations made for them. Whereas for a physical understanding of the atmosphere the challenge lies in the choice of profound approximations to get simple but still realistic solutions (or better say: simple formulae). In contrast for the testing of dynamical cores as few as possible approximations should be made, or better say, the system of equations used for the analytic solution should be as close as possible to the equations underlying the numerical model. It is not important that the analytic solution (or formula) is simple, but that it can be calculated with a much higher numerical confidence than the numerical solution of PDE's, e.g. by numerically calculating integrals or Fouriertransforms.

During the COSMO priority project 'Runge-Kutta' a program for calculating linear solutions for flow over mountains of the compressible Euler-equations for a stably stratified atmosphere with a constant Brunt-Vaisala frequency was developed. The prerequisites are: adiabasy, no friction, no Coriolis force, dry atmosphere, and no earth curvature. Further on the prerequisite is made, that the inflow does not change direction with height. The only a priori approximation done is that of linearisation about the mountain height. These prerequisites are easily fulfilled for the dynamical core by switching off the other processes; the approximation is easily fulfilled by choosing a very small hill. The only additional (a posteriori) approximation is to neglect a small height dependency of the vertical wavelength $k_{z}$. The error induced by this is estimated by the program.

All in all the requirements are stronger than those which can be found in the literature, but this allows to reduce the number of approximations only to the two mentioned before, which can be easily controlled.

## 2 Linearised equations

This section presents the derivation of the solution for linear flow over mountains. The starting point are the papers of Smith $(1979,1980)$ with the following prerequesites:

- no friction
- only adiabatic processes (in particular no phase changes)
- ideal gas law
- $c_{p}=$ const.,$c_{v}=$ const.,$R=$ const.
- All movements are taking place on a plane (no earth curvature)
- no Coriolis force

This leads to the following system of equations (Smith, 1979):

$$
\begin{align*}
\rho \frac{d u}{d t} & =-\frac{\partial p}{\partial x},  \tag{1}\\
\rho \frac{d v}{d t} & =-\frac{\partial p}{\partial y},  \tag{2}\\
\delta_{1} \rho \frac{d w}{d t} & =-\frac{\partial p}{\partial z}-\rho g,  \tag{3}\\
\delta_{2} \frac{\partial \rho}{\partial t}+\delta_{3} \boldsymbol{v} \cdot \nabla \rho+\rho \nabla \cdot \boldsymbol{v} & =0,  \tag{4}\\
\frac{d p}{d t} & =c^{2} \frac{d \rho}{d t}, \quad c^{2}:=\frac{c_{p}}{c_{V}} \frac{p}{\rho},  \tag{5}\\
p & =\rho R T . \tag{6}
\end{align*}
$$

Here some tracer-parameters were introduced:

- $\delta_{1}=0 / 1$ : hydrostatic / non-hydrostatic approximation
- $\delta_{2}=0 / 1$ : incompressible / compressible model
- $\delta_{3}=0 / 1$ : shallow / deep atmosphere

To linearize these equations a base state has to be chosen: it has to be stationary, hydrostatic and at most dependent from $z$ (the last choice requires the neglection of the Coriolis force)

$$
\begin{align*}
u_{0} & =u_{0}(z),  \tag{7}\\
v_{0} & =0,  \tag{8}\\
w_{0} & =0,  \tag{9}\\
T_{0} & =T_{0}(z),  \tag{10}\\
p_{0} & =\rho_{0} R T_{0},  \tag{11}\\
\frac{\partial p_{0}}{\partial z} & =-g \rho_{0} . \tag{12}
\end{align*}
$$

Later on we will consider an atmosphere with a constant Brunt-Vaisala frequency $N$. This leads to the base state temperature profile

$$
\begin{align*}
T_{0}(z) & =T_{0}(z=0)\left(a-(a-1) e^{z / H}\right),  \tag{13}\\
H & :=\frac{g}{N^{2}} \sim 100 \mathrm{~km}  \tag{14}\\
a & :=\frac{g^{2}}{N^{2} c_{p} T_{0}(z=0)} \sim 3 . \tag{15}
\end{align*}
$$

Such an atmosphere has negative values of temperature above $z_{\max }=H \log a /(a-1) \sim 35 \mathrm{~km}$ for realistic values.

Perturbation equations The above chosen base state leads to the perturbation equations

$$
\begin{align*}
\rho_{0}\left(\frac{\partial u^{\prime}}{\partial t}+u_{0} \frac{\partial u^{\prime}}{\partial x}+w^{\prime} \frac{\partial u_{0}}{\partial z}\right) & =-\frac{\partial p^{\prime}}{\partial x},  \tag{16}\\
\rho_{0}\left(\frac{\partial v^{\prime}}{\partial t}+u_{0} \frac{\partial v^{\prime}}{\partial x}\right) & =-\frac{\partial p^{\prime}}{\partial y},  \tag{17}\\
\rho_{0}\left(\delta_{1} \frac{\partial w^{\prime}}{\partial t}+\delta_{1} u_{0} \frac{\partial w^{\prime}}{\partial x}\right) & =-\frac{\partial p^{\prime}}{\partial z}-g \rho^{\prime},  \tag{18}\\
\delta_{2} \frac{\partial \rho^{\prime}}{\partial t}+\delta_{3} u_{0} \frac{\partial \rho^{\prime}}{\partial x}+\delta_{3} w^{\prime} \frac{\partial \rho_{0}}{\partial z} & =-\rho_{0}\left(\frac{\partial u^{\prime}}{\partial x}+\frac{\partial v^{\prime}}{\partial y}+\frac{\partial w^{\prime}}{\partial z}\right),  \tag{19}\\
\frac{\partial p^{\prime}}{\partial t}+\delta_{4} u_{0} \frac{\partial p^{\prime}}{\partial x}+\underbrace{w^{\prime} \frac{\partial p_{0}}{\partial z}}_{=-g \rho_{0} w^{\prime}} & =c_{0}^{2}\left(\frac{\partial \rho^{\prime}}{\partial t}+u_{0} \frac{\partial \rho^{\prime}}{\partial x}+w^{\prime} \frac{\partial \rho_{0}}{\partial z}\right) . \tag{20}
\end{align*}
$$

where a 4th tracer-parameter was introduced (Pichler, 1997):

- $\delta_{4}=0 / 1$ : small / big Mach-numbers.

These perturbation equations are fouriertransformed, i.e. the fields are represented by waves of the form

$$
\begin{equation*}
\phi^{\prime}(x, y, z, t)=\phi^{\prime}\left(k_{x}, k_{y}, z, \omega\right) \cdot e^{i\left(k_{x} x+k_{y} y-\omega t\right)} . \tag{21}
\end{equation*}
$$

This leads to

$$
\begin{align*}
-i \omega u^{\prime}+i k_{x} u_{0} u^{\prime}+\frac{\partial u_{0}}{\partial z} w^{\prime} & =-i k_{x} \frac{1}{\rho_{0}} p^{\prime},  \tag{22}\\
-i \omega v^{\prime}+i k_{x} u_{0} v^{\prime} & =-i k_{y} \frac{1}{\rho_{0}} p^{\prime},  \tag{23}\\
\delta_{1}\left(-i \omega w^{\prime}+i k_{x} u_{0} w^{\prime}\right) & =-\frac{1}{\rho_{0}} \frac{\partial p^{\prime}}{\partial z}-\frac{\rho^{\prime}}{\rho_{0}} g,  \tag{24}\\
-\delta_{2} i \omega \rho^{\prime}+\delta_{3} i k_{x} u_{0} \rho^{\prime}+\delta_{3} \frac{\partial \rho_{0}}{\partial z} w^{\prime} & =-\rho_{0}\left(i k_{x} u^{\prime}+i k_{y} v^{\prime}+\frac{\partial w^{\prime}}{\partial z}\right),  \tag{25}\\
-i \omega p^{\prime}+\delta_{4} i k_{x} u_{0} p^{\prime}-g \rho_{0} w^{\prime} & =c_{0}^{2}\left(-i \omega \rho^{\prime}+i k_{x} u_{0} \rho^{\prime}+\frac{\partial \rho_{0}}{\partial z} w^{\prime}\right) . \tag{26}
\end{align*}
$$

We first express $u^{\prime}, v^{\prime}$ and $\rho^{\prime}$ through the other variables

$$
\begin{align*}
u^{\prime} & =\frac{1}{\omega-k_{x} u_{0}}\left(k_{x} \frac{1}{\rho_{0}} p^{\prime}-i \frac{\partial u_{0}}{\partial z} w^{\prime}\right)  \tag{27}\\
v^{\prime} & =\frac{1}{\omega-k_{x} u_{0}}\left(k_{y} \frac{1}{\rho_{0}} p^{\prime}\right)  \tag{28}\\
\rho^{\prime} & =\frac{1}{\omega-k_{x} u_{0}}\left(\frac{1}{c_{0}^{2}}\left(\omega-\delta_{4} k_{x} u_{0}\right) p^{\prime}-i \rho_{0}\left(\frac{g}{c_{0}^{2}}+\frac{1}{\rho_{0}} \frac{\partial \rho_{0}}{\partial z}\right) w^{\prime}\right) . \tag{29}
\end{align*}
$$

It is common practice to introduce the following denotations:
Heterogenity (Queney, 1947)

$$
\begin{equation*}
S_{0}:=\frac{1}{\rho_{0}} \frac{d \rho_{0}}{d z} \equiv \frac{d \log \rho_{0}}{d z} \tag{30}
\end{equation*}
$$

## Stability parameter

$$
\begin{equation*}
\beta_{0}:=\frac{1}{\Theta_{0}} \frac{d \Theta_{0}}{d z} \equiv \frac{d \log \Theta_{0}}{d z} \tag{31}
\end{equation*}
$$

## Mach-number

$$
\begin{equation*}
M a:=\frac{u_{0}}{c_{0}} \tag{32}
\end{equation*}
$$

We are interested only in the stationary case $\omega=0$ and to begin with we consider the case $k_{x} \neq 0$.
(29) simplifies to

$$
\begin{equation*}
\rho^{\prime}=\frac{1}{c_{0}^{2}} \delta_{4} p^{\prime}-i \rho_{0} \frac{1}{k_{x} u_{0}} \beta_{0} w^{\prime} \tag{33}
\end{equation*}
$$

and (26) leads to

$$
\begin{equation*}
p^{\prime}=i \frac{k_{x}}{k_{x}^{2}+k_{y}^{2}} \frac{\rho_{0} u_{0}}{\mu_{0}}\left(\left[\delta_{3} \frac{g}{c_{0}^{2}}+\frac{1}{u_{0}} \frac{\partial u_{0}}{\partial z}\right] w^{\prime}-\frac{\partial w^{\prime}}{\partial z}\right) \tag{34}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu_{0}:=1-\delta_{3} \delta_{4} \frac{k_{x}^{2}}{k_{x}^{2}+k_{y}^{2}} M a^{2} \tag{35}
\end{equation*}
$$

was defined. Both inserted into (24) delivers an ODE of 2 nd order for $w^{\prime}\left(k_{x}, k_{y}, z, \omega\right)$

$$
\begin{equation*}
\frac{d^{2} w^{\prime}}{d z^{2}}+\frac{d w^{\prime}}{d z}\left(\frac{d}{d z} \log d(z)\right)+b(z) w^{\prime}=0 \tag{36}
\end{equation*}
$$

with

$$
\begin{align*}
d(z)= & \frac{\partial}{\partial z} \log \frac{\rho_{0}}{\mu_{0}}+\left(\delta_{4}-\delta_{3}\right) \frac{g}{c_{0}^{2}}=\left\{\begin{array}{cl}
\frac{\rho_{0}}{\mu_{0}} & , \text { if } \delta_{4}=\delta_{3} \\
\frac{1}{\Theta_{0} \mu_{0}} & , \text { if } \delta_{4}=1, \delta_{3}=0 \\
\frac{\Theta_{0} \rho_{0}^{2}}{\mu_{0}} & , \text { if } \delta_{4}=0, \delta_{3}=1
\end{array}\right.  \tag{37}\\
b(z)= & -\delta_{1} \mu_{0} k_{h}^{2}-\left[\delta_{3} \frac{g}{c_{0}^{2}}+\frac{\partial}{\partial z} \log u_{0}\right]\left(\frac{\partial}{\partial z} \log \frac{\rho_{0} u_{0}}{\mu_{0}}+\delta_{4} \frac{g}{c_{0}^{2}}\right) \\
& -\left[\delta_{3} \frac{\partial}{\partial z} \frac{g}{c_{0}^{2}}+\frac{\partial^{2}}{\partial z^{2}} \log u_{0}\right]+\frac{g \beta_{0}}{u_{0}^{2}} \mu_{0} \frac{k_{h}^{2}}{k_{x}^{2}} \tag{38}
\end{align*}
$$

With the variable transformation

$$
\begin{equation*}
w^{\prime}(z)=\frac{1}{\sqrt{d}} W(z) \tag{39}
\end{equation*}
$$

this can be transformed into an 'oscillation equation'

$$
\begin{align*}
\frac{d^{2} W}{d z^{2}}+k_{z}^{2} W & =0  \tag{40}\\
k_{z}^{2}\left(k_{x}, k_{y}\right) & :=\frac{1}{4} \frac{d^{\prime 2}}{d^{2}}-\frac{1}{2} \frac{d^{\prime \prime}}{d}+b \tag{41}
\end{align*}
$$

Therefore the main task is to solve this ODE with the appropriate boundary conditions, then to calculate the fields $w^{\prime}, p^{\prime}, u^{\prime}, v^{\prime}$, and $\rho^{\prime}$ by the above given formulas and to carry out a Fourier backtransformation to get these fields in the physical space.
There remain two special cases. The first one is $\omega=0, k_{x}=0, k_{y} \neq 0$. The original fouriertransformed perturbation equations (22)-(26) deliver successively $p^{\prime}=0, \rho^{\prime}=0$, $w^{\prime}=0$, and $v^{\prime}=0$. It is remarkable that no statement can be given for $u^{\prime}$. This describes the fact that in a frictionless flow over a flat plane an arbitrary vertical shear can occur. The second special case is $\omega=0, k_{x}=0, k_{y}=0$. Now $u^{\prime}$ and $v^{\prime}$ can be chosen arbitrarily. $p^{\prime}$ and $\rho^{\prime}$ are connected by (24). In the case $\delta_{3}=0$ (shallow atmosphere approximation) it follows from the lower boundary condition (see below) $w^{\prime}=0$. In the case $\delta_{3}=1$ (deep atmosphere) also only $w^{\prime}=0$ leads to an equation system without contradictions.

Boundary conditions We prescribe an orography $h(x, y)$. The lower boundary condition consists in the free-slip condition. Its linearisation in $z=h(x, y)$ leads to

$$
\begin{equation*}
w^{\prime}(x, y, z=0) \approx U_{0} \frac{\partial h}{\partial x} \tag{42}
\end{equation*}
$$

A horizontal Fourier transform delivers

$$
\begin{equation*}
\tilde{w}^{\prime}\left(k_{x}, k_{y}, z=0\right) \approx i k_{x} U_{0} h\left(k_{x}, k_{y}\right) \tag{43}
\end{equation*}
$$

and with the above variable transformation follows

$$
\begin{equation*}
W\left(k_{x}, k_{y}, z=0\right) \approx i k_{x} U_{0} h\left(k_{x}, k_{y}\right) \sqrt{d(z=0)} \tag{44}
\end{equation*}
$$

The upper boundary condition is a little bit more delicate. We assume that $k_{z}^{2}$ does not depend from $z$. In this case we can solve the oscillation equation directly

$$
\begin{equation*}
W\left(k_{x}, k_{y}, z\right)=A e^{i k_{z} z}+B e^{-i k_{z} z} \tag{45}
\end{equation*}
$$

Two cases have to be distinguished:

- case $k_{z}^{2}<0$ : only a solution which decays with height seems to be physical. We define $k_{z}:=i \sqrt{-k_{z}^{2}}$ and omit the term $\sim B$.
- case $k_{z}^{2}>0$ : the reuirement is that no energy transport to the ground takes place. Again we omit the term $\sim B$ and define $k_{z}:=\operatorname{sgn}\left(U_{0} k_{x}\right) \cdot \sqrt{k_{z}^{2}}$ (Smith, 1980).

Therefore for height independent $k_{z}^{2}$ we get the solution

$$
\begin{equation*}
W\left(k_{x}, k_{y}, z\right)=i k_{x} U_{0} h\left(k_{x}, k_{y}\right) \sqrt{d\left(k_{x}, k_{y}, z=0\right)} \cdot e^{i k_{z} z} \tag{46}
\end{equation*}
$$

## 3 A Case Study

A first test with the COSMO-model was done with the case described in Schär et al. (2002)[section 5b]. A 2D-flow over a modulated Gaussian hill

$$
\begin{equation*}
h(x)=h_{0} e^{-\frac{x^{2}}{b^{2}}} \cos ^{2} \pi \frac{x}{\lambda} \tag{47}
\end{equation*}
$$

with $b=5 \mathrm{~km}$ and $\lambda=4 \mathrm{~km}$ is considered. The atmospheric conditions are: inflow velocity of $U_{0}=10 \mathrm{~m} / \mathrm{s}$, a constant Brunt-Vaisala-frequency $N=0.011 / \mathrm{s}$, and a surface temperature of $T(z=0)=288 \mathrm{~K}$. The maximum height of the mountains is $h_{0}=25 \mathrm{~m}$ (reduced by a factor of 10 compared to Schär et al., 2002). This results in a small inverse vertical Froude number of $1 / F r=N h_{0} / U_{0}=0.025$ and therefore allows the application of a linearized solution.

Simulations with two resolutions were made: the first uses $\Delta x=500 \mathrm{~m}, \Delta z=300 \mathrm{~m}$ with a time step of $\Delta t=8 \mathrm{~s}$ as in Schär et al. (2002), the second uses $\Delta x=250 \mathrm{~m}, \Delta z=200 \mathrm{~m}$, and therefore a slightly smaller time step of $\Delta t=6 \mathrm{~s}$. The first setup uses 80 vertical levels, the second one 120 levels, therefore in both setups the upper model boundary lies in $z=24 \mathrm{~km}$. The upper relaxation zone starts in $z=13 \mathrm{~km}$ with a thickness of 11 km . Such a thick relaxation zone is crucial to damp out perturbations to be able to properly compare with the analytical solution. The COSMO-version 4.6 was used with the Runge-Kutta dynamical core (Namelist-Parameters irunge_kutta=1, irk_order=3). Simulation results after 24 h and the appropriate analytic solution are shown in figure. The similarity with the analytical solution


Figure 1: Comparison of the vertical velocity $w$ between the COSMO-model (coloured) and the analytic solution (black lines) for the Schär et al. (2002) test case. Above: $\Delta x=500 \mathrm{~m}, \Delta z=300 \mathrm{~m}$, below: $\Delta x=250 \mathrm{~m}, \Delta z=200 \mathrm{~m}$.
is very close for both resolutions. Therefore even with $\Delta x=500 \mathrm{~m}$, a good convergence has been reached.

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