The Coordinate Transformations of the 3-Dimensional Turbulent Diffusion in LMK

M. Baldauf

Deutscher Wetterdienst, Kaiserleistr. 42, D-63067 Offenbach a.M., Germany

1 Introduction

At the Deutscher Wetterdienst the numerical weather forecast model LMK (LM-Kürzestfrist) is currently under development. It is based on the LM (Lokal Modell) and shall be used for very short range forecasts (up to 18 hours) and with a resolution on the meso-γ-scale (about 2.8 km). Among other new parameterizations, e.g. a 6-class cloud microphysics scheme with graupel and a shallow convection parameterization, LMK shall contain a 3-dimensional (3D) turbulence model instead of the 1-dimensional column physics designed for the current LM resolution (Raschendorfer 2004). An appropriate 3D turbulence formulation is already contained in LLM, the large eddy simulation version of LM (Herzog, 2002a, and Herzog, 2002b) which was transferred into LMK (Förstner, 2005). Despite the fact, that it is not yet clear if a 3D turbulence is really required at the meso-γ-scale, for further applications of the model with increasing horizontal resolutions a 3D formulation of turbulence will surely be necessary.

The LLM was mainly designed for very small scale climatology studies with horizontal resolutions of about $\Delta x \sim 100$ m and therefore it uses cartesian coordinates instead of terrain following coordinates required for regional numerical weather prediction models. As a first step, this work derives the terms associated with terrain following coordinates both for the fluxes of momentum and arbitrary scalars and for the divergence of these fluxes. In principle, these terms were already derived in the LM documentation (Doms and Schättler, 2002; Doms et.al., 2005), but an error was found in their derivation: especially the divergence of a vector and a second-order tensor was handled identical there, which is not correct. Consequently a new derivation of these terms is presented here.

2 The flux divergences

To derive the coordinate transformation of the diffusion terms, in the equation of motion we only consider the time derivative of the velocity components and the divergence of the momentum fluxes

$$\frac{\partial v^i}{\partial t} = -\frac{1}{\rho}\nabla_j T^{ij} = -\frac{1}{\rho}\left(\frac{\partial}{\partial x^j}T^{ij} + \Gamma^i_{jk}T^{kj} + \Gamma^j_{jk}T^{ik}\right). \tag{1}$$

This equation is valid in *every arbitrary* coordinate system (e.g. Stephani, 1988). T^{ij} are the 2-fold contravariant components of the momentum flux tensor, v^i are the contravariant components of the velocity vector. Γ^k_{ij} are the Christoffel-symbols of 2. kind, defined by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kh}\left(g_{ih,j} + g_{jh,i} - g_{ij,h}\right). \tag{2}$$

g^{ij} is the metric tensor; partial derivatives are denoted by a comma ','.

A special quality of the LM coordinate system consists in the fact, that on the one hand it is spherical and terrain-following, but on the other hand it uses base vectors of a purely spherical coordinate system. Beyond this, not the contravariant components are used, but the so called physical components (i.e. they are related to normalized base vectors). Because of that, the elegant and compact formulation in (1) is lost, unfortunately.

To express equation (1) in the special LM coordinate system, it turns out to be convenient to apply it in purely spherical coordinates $x^i = (r, \lambda, \phi)$. This reflects on the one hand the usage of the spherical base vectors. On the other hand, the metric tensor and the Christoffel symbols are well known for spherical coordinates and can be found in many text books (see also Appendix A) instead of deriving them for the fully spherical and terrain-following coordinates.

Starting from Eq. (1) we first introduce physical components. Here we denote them with a star $*$, in contrast to the LM-documentation (Doms et.al., 2005), e.g.:

$$v^{*i} = \sqrt{g_{(ii)}}v^i. \tag{3}$$

A bracket (..) around indices means, that the summation convention must not be applied to them. A summation is only carried out if there is also a double occurence of the same indices without brackets, so that they only 'run in common' (one should not confuse this with symmetrising brackets, partially used in the literature).

From Eq. (1) follows

$$\rho\frac{\partial v^{*i}}{\partial t} = \rho\sqrt{g_{(ii)}}\frac{\partial v^i}{\partial t} = -\sqrt{g_{(ii)}}\left[\frac{\partial}{\partial x^j}\left(T^{*ij}\frac{1}{\sqrt{g_{(ii)}}}\frac{1}{\sqrt{g_{(jj)}}}\right) + +\Gamma^i_{jk}T^{*kj}\frac{1}{\sqrt{g_{(jj)}}}\frac{1}{\sqrt{g_{(kk)}}} + \Gamma^j_{jk}T^{*ik}\frac{1}{\sqrt{g_{(ii)}}}\frac{1}{\sqrt{g_{(kk)}}}\right]. \tag{4}$$

The transition to terrain following coordinates (denoted with x'^i) only means an application of the chain rule in the first term on the right side. Simultaneous application of the product rule delivers

$$\begin{aligned}\rho\frac{\partial v^{*i}}{\partial t} = -\sqrt{g_{(ii)}}\Bigg[&\frac{1}{\sqrt{g_{(ii)}}}\frac{1}{\sqrt{g_{(jj)}}}\frac{\partial x'^l}{\partial x^j}\frac{\partial}{\partial x\prime l}T^{*ij}\\ &+T^{*ij}\left(\frac{1}{\sqrt{g_{(jj)}}}\frac{\partial}{\partial x^j}\frac{1}{\sqrt{g_{(ii)}}} + \frac{1}{\sqrt{g_{(ii)}}}\frac{\partial}{\partial x^j}\frac{1}{\sqrt{g_{(jj)}}}\right)\\ &+\Gamma^i_{jk}T^{*kj}\frac{1}{\sqrt{g_{(jj)}}}\frac{1}{\sqrt{g_{(kk)}}} + \Gamma^j_{jk}T^{*ik}\frac{1}{\sqrt{g_{(ii)}}}\frac{1}{\sqrt{g_{(kk)}}}\Bigg].\end{aligned} \tag{5}$$

Derivatives of the metric tensor can be expressed by Christoffel-symbols (by solving Eq. (2)) to derive the equation

$$\frac{\partial}{\partial x^j}\frac{1}{\sqrt{g_{(ii)}}} = -\frac{1}{2\sqrt{g_{(ii)}}^3}\, g_{(ii),j} = -\frac{1}{\sqrt{g_{(ii)}}^3}\, g_{k(i)}\,\Gamma^k_{(i)j}. \tag{6}$$

It follows

$$\begin{aligned}\rho\frac{\partial v^{*i}}{\partial t} &= \underbrace{-\frac{1}{\sqrt{g_{(jj)}}}\frac{\partial x'^l}{\partial x^j}\frac{\partial}{\partial x'^l}T^{*ij}}_{(a)} \underbrace{+T^{*ij}\frac{1}{\sqrt{g_{(jj)}}}\frac{1}{g_{(ii)}}g_{k(i)}\Gamma^k_{(i)j}}_{(b)} \underbrace{+T^{*ij}\frac{1}{\sqrt{g_{(jj)}}^3}g_{k(j)}\Gamma^k_{(j)j}}_{(c)} \\ &\quad \underbrace{-\Gamma^i_{jk}T^{*kj}\frac{1}{\sqrt{g_{(jj)}}}\frac{1}{\sqrt{g_{(kk)}}}\sqrt{g_{(ii)}}}_{(d)} \underbrace{-\Gamma^j_{jk}T^{*ik}\frac{1}{\sqrt{g_{(kk)}}}}_{(e)}.\end{aligned} \tag{7}$$

This equation generally describes the temporal change and the diffusion term of the physical components refered to a (normalized) base (in a coordinate system x^j), where all the fields are defined in a coordinate system x'^j.

Now we will specify the LM coordinate system. In Appendix A the metric tensor and Christoffel-symbols for spherical coordinates x^i are given; in Appendix B the transformation to terrain-following coordinates x'^i. Equations (46), (47) and (49) have to be inserted in Eq. (7) and after cumbersome, but uncomplicated calculation one finally arrives at

$$\begin{aligned}\rho\frac{\partial u^*}{\partial t} &= -\frac{1}{r\cos\phi}\frac{\partial T^{*11}}{\partial\lambda} - \frac{J_\lambda}{\sqrt{G}}\frac{1}{r\cos\phi}\frac{\partial T^{*11}}{\partial\zeta} - \frac{1}{r}\frac{\partial T^{*12}}{\partial\phi} - \frac{J_\phi}{\sqrt{G}}\frac{1}{r}\frac{\partial T^{*12}}{\partial\zeta} + \frac{1}{\sqrt{G}}\frac{\partial T^{*13}}{\partial\zeta} \\ &\quad -\frac{3}{r}\,T^{*13} + \frac{2\tan\phi}{r}\,T^{*12},\end{aligned} \tag{8}$$

$$\begin{aligned}\rho\frac{\partial v^*}{\partial t} &= -\frac{1}{r\cos\phi}\frac{\partial T^{*12}}{\partial\lambda} - \frac{J_\lambda}{\sqrt{G}}\frac{1}{r\cos\phi}\frac{\partial T^{*12}}{\partial\zeta} - \frac{1}{r}\frac{\partial T^{*22}}{\partial\phi} - \frac{J_\phi}{\sqrt{G}}\frac{1}{r}\frac{\partial T^{*22}}{\partial\zeta} + \frac{1}{\sqrt{G}}\frac{\partial T^{*23}}{\partial\zeta} \\ &\quad -\frac{3}{r}\,T^{*23} - \frac{\tan\phi}{r}\,(T^{*11} - T^{*22}),\end{aligned} \tag{9}$$

$$\begin{aligned}\rho\frac{\partial w^*}{\partial t} &= -\frac{1}{r\cos\phi}\frac{\partial T^{*13}}{\partial\lambda} - \frac{J_\lambda}{\sqrt{G}}\frac{1}{r\cos\phi}\frac{\partial T^{*13}}{\partial\zeta} - \frac{1}{r}\frac{\partial T^{*23}}{\partial\phi} - \frac{J_\phi}{\sqrt{G}}\frac{1}{r}\frac{\partial T^{*23}}{\partial\zeta} + \frac{1}{\sqrt{G}}\frac{\partial T^{*33}}{\partial\zeta} \\ &\quad -\frac{2}{r}\,T^{*33} + \frac{1}{r}\,(T^{*11} + T^{*22}) + \frac{\tan\phi}{r}T^{*23}.\end{aligned} \tag{10}$$

Now the diffusion terms for scalar quantities s with an appropriate diffusion flux H^i shall be derived:

$$\frac{\partial s}{\partial t} = -\nabla_j H^j = -\frac{\partial}{\partial x^j}H^j - \Gamma^j_{jk}H^k. \tag{11}$$

The derivation is analogous to the former, but now some terms can be neglected from the starting point and it remains

$$\frac{\partial s}{\partial t} = -\frac{1}{\sqrt{g_{(jj)}}}\frac{\partial x'^l}{\partial x^j}\frac{\partial}{\partial x'^l}H^{*j} + \frac{1}{\sqrt{g_{(jj)}}^3}g_{k(j)}\Gamma^k_{(j)j}H^{*j} - \Gamma^j_{jk}\frac{1}{\sqrt{g_{(kk)}}}H^{*k}. \tag{12}$$

This corresponds obviously to the terms (a), (c) and (e) in Eq. (7). By insertion of the terms from Appendices A and B, it follows

$$\begin{aligned}\frac{\partial s}{\partial t} &= -\frac{1}{r\cos\phi}\frac{\partial H^{*1}}{\partial\lambda} - \frac{J_\lambda}{\sqrt{G}}\frac{1}{r\cos\phi}\frac{\partial H^{*1}}{\partial\zeta} - \frac{1}{r}\frac{\partial H^{*2}}{\partial\phi} - \frac{J_\phi}{\sqrt{G}}\frac{1}{r}\frac{\partial H^{*2}}{\partial\zeta} + \frac{1}{\sqrt{G}}\frac{\partial H^{*3}}{\partial\zeta} \\ &\quad -\frac{2}{r}\,H^{*3} + \frac{\tan\phi}{r}\,H^{*2}.\end{aligned} \tag{13}$$

The single terms of this equation can be interpreted descriptive. The 1st, 3rd, and 5th term in the first line are corresponding completely to the x-, y-, and z-derivative in the calculation of the divergence in cartesian coordinates:

$$\frac{1}{r\cos\phi}\frac{\partial H^{*1}}{\partial\lambda} = \left.\frac{\partial H^{*1}}{\partial x}\right|_{\phi,\zeta}, \tag{14}$$

$$\frac{1}{r}\frac{\partial H^{*2}}{\partial\phi} = \left.\frac{\partial H^{*2}}{\partial y}\right|_{\lambda,\zeta}, \tag{15}$$

$$-\frac{1}{\sqrt{G}}\frac{\partial H^{*3}}{\partial\zeta} = \left.\frac{\partial H^{*3}}{\partial z}\right|_{\lambda,\phi}. \tag{16}$$

The 2nd and 4th terms give a correction from the terrain-following coordinate system, which obviously is given alone from the ζ-derivative

$$\frac{J_\lambda}{\sqrt{G}}\frac{1}{r\cos\phi}\frac{\partial H^{*1}}{\partial\zeta} = \frac{\left.\frac{\partial z}{\partial\lambda}\right|_\zeta}{-\frac{\partial z}{\partial\zeta}}\frac{1}{r\cos\phi}\frac{\partial H^{*1}}{\partial\zeta} = -\left.\frac{\partial z}{\partial x}\right|_\zeta\cdot\left.\frac{\partial H^{*1}}{\partial z}\right|_{\lambda,\phi}, \tag{17}$$

$$\frac{J_\phi}{\sqrt{G}}\frac{1}{r}\frac{\partial H^{*2}}{\partial\zeta} = \frac{\left.\frac{\partial z}{\partial\phi}\right|_\zeta}{-\frac{\partial z}{\partial\zeta}}\frac{1}{r}\frac{\partial H^{*2}}{\partial\zeta} = -\left.\frac{\partial z}{\partial y}\right|_\zeta\cdot\left.\frac{\partial H^{*2}}{\partial z}\right|_{\lambda,\phi}. \tag{18}$$

One has to obey, that the relations (14) - (18) are valid only on a local tangential plane (with $dx = r\cos\phi\, d\lambda$, and $dy = r\, d\phi$) and are presented here only for illustration purposes. Analogous relations follow for the momentum flux divergences (8)-(10) above.

The two terms in the second line of (13) are given by the spherical basis and can be illustrated in the following manner. We prescribe a constant (in the spherical base) scalar flux field $H^{*i} = const. = (0, 0, h)$, that means a radially outside directed vector field with constant absolute value. Starting from the integral form of the balance equation (11)

$$\int_V \frac{\partial s}{\partial t} dV = -\int_{\partial V} \mathbf{H}\cdot\mathbf{d}\sigma \tag{19}$$

and choosing an infinitesimal spherical segment as integration volume

$$dV = d\sigma\cdot dr \tag{20}$$

with the surface element

$$d\sigma = (r\, d\phi)\cdot(r\cos\phi\, d\lambda) \tag{21}$$

it follows

$$\frac{\partial s}{\partial t}\cos\phi\, d\phi\, d\lambda\, r^2 dr = -H^{*3}\left[((r+dr)d\phi)\cdot((r+dr)\cos\phi\, d\lambda) - (r\, d\phi)\cdot(r\cos\phi\, d\lambda)\right] \tag{22}$$

and finally by expansion

$$\frac{\partial s}{\partial t} = -H^{*3}\frac{2}{r} + \ldots . \tag{23}$$

Therefore this correction term follows from the divergence of the radial base vectors in the spherical coordinate system. Obviously, this term is not contained in the original derivation in Doms, et.al., 2005, [Eq. (3.3)].

Analogously the 2nd term in the second line of Eq. (13) can be explained by the convergence (on the northern hemisphere) of the meridional base vectors. To see this, one chooses $H^{*i} = const. = (0, h, 0)$ and the same infinitesimal volume element; the flux divergence now arises from the lateral boundary surfaces (this term also arises in Doms, et.al., 2005, [Eq. (3.3)], but without the factor 2). This also explains, why there are no such terms for the H^{*1}-component: the zonal base vectors do not possess such a convergence/divergence.

The same considerations can be made for the equations of motion (8)-(10). Again, the terms in the second line arise by the spherical basis due to the curvature of the earth. Nevertheless the vectorial character of these equations complicates a quantitative illustration compared to the scalar equation.

How strong is the practical relevance of these terms? Let us first look to the terms which are only addressed to the earth curvature. In the equations of motion, we have terms of the form $T^{*ij}/r/\rho$. For rough estimations we can set $T^{*ij}/\rho \sim TKE$, where we can limit $TKE < 10\ \mathrm{m}^2/\mathrm{s}^2$; this occurs in nearly neutral boundary layers with very strong winds. Now we can estimate (where we take the biggest occuring coefficient of 3)

$$\max \frac{3|T^{*ij}|}{r\rho} \sim \max 3 \frac{TKE}{R_E} \sim 3 \frac{10\, m^2/s^2}{6 \cdot 10^6\, m} \sim 0.5 \cdot 10^{-5} \frac{m}{s^2} \tag{24}$$

(R_E = earth radius). Even this extreme case is more than one order less than a Coriolis force for small wind velocities with $2\Omega \times v \sim 10^{-4} s^{-1} \cdot 1\, m/s \sim 10^{-4}\, m/s^2$ and therefore such terms can be neglected.

In the equation for the scalar flux divergence, we roughly estimate $H^{*i} \sim |s| \cdot \sqrt{TKE}$ and find

$$\max \frac{|H^{*i}|}{r} \sim \max |s| \cdot \frac{\max \sqrt{TKE}}{R_E} \sim \max |s| \cdot \frac{3\, m/s}{6 \cdot 10^6\, m} \sim \frac{\max |s|}{2 \cdot 10^6 s} \sim \frac{\max |s|}{20\, d} \tag{25}$$

These terms mean at most a very slow change of the scalar variable in the order of about 20 days and therefore can be neglected, too.

Next, we look at the terms connected with the terrain-following coordinate. At least near the bottom, we can estimate for the typical LMK resolution and area (Germany and the most part of the Alpines) a maximum terrain slope of

$$\max \left| \frac{\partial z}{\partial x} \right| \approx \max \frac{|\Delta h|}{\Delta x} \approx \frac{1000\, m}{2800\, m} \approx 0.3 \tag{26}$$

Normally, the vertical part of the flux divergences are stronger than the horizontal parts, therefore, as can be seen from Eq. (17), terms connected with the terrain-following coordinate cannot be neglected.

3 The transformation of the turbulence closure

For calculating the momentum fluxes we start from a formulation with a scalar eddy viscosity K, which connects the fluxes with the deformation tensor D^{ij} in a gradient ansatz:

$$T^{ij} = -\rho K D^{ij} = -\rho K (g^{il} \nabla_l v^j + g^{jl} \nabla_l v^i). \tag{27}$$

Often, a diagonal term $2/3 g^{ij} E$, where E is the turbulent kinetic energy, is added to the right hand side. Such a term guaranties consistency for an incompressible (and therefore divergence free) medium; this can be seen by taking the trace of (27). We neglect it here, because it is not used in Herzog, 2002a. But because of its simple form (E is a scalar), it is not affected by the following transformations and can be implemented directly. Using the physical components delivers

$$\frac{T^{*ij}}{\sqrt{g_{(ii)}}\sqrt{g_{(jj)}}} = -\rho K \left[g^{il}\frac{\partial}{\partial x^l}\left(\frac{v^{*j}}{\sqrt{g_{(jj)}}}\right) + g^{il}\Gamma^j_{lk}\frac{v^{*k}}{\sqrt{g_{(kk)}}} + g^{jl}\frac{\partial}{\partial x^l}\left(\frac{v^{*i}}{\sqrt{g_{(ii)}}}\right) + g^{jl}\Gamma^i_{lk}\frac{v^{*k}}{\sqrt{g_{(kk)}}}\right] \tag{28}$$

and introduction of terrain-following coordinates, application of the product rule and Eq. (6) yields

$$\begin{aligned} T^{*ij} = \quad & -\rho K \left[\sqrt{g_{(ii)}} g^{il} \frac{\partial x'^m}{\partial x^l}\frac{\partial v^{*j}}{\partial x'^m} - \frac{g^{il}}{g_{(jj)}}\sqrt{g_{(ii)}} g_{k(j)}\Gamma^k_{(j)l} v^{*j} + \frac{g^{il}}{\sqrt{g_{(kk)}}}\sqrt{g_{(ii)}}\sqrt{g_{(jj)}}\Gamma^j_{lk}v^{*k} \right. \\ & \left. \sqrt{g_{(jj)}} g^{jl} \frac{\partial x'^m}{\partial x^l}\frac{\partial v^{*i}}{\partial x'^m} - \frac{g^{jl}}{g_{(ii)}}\sqrt{g_{(jj)}} g_{k(i)}\Gamma^k_{(i)l} v^{*i} + \frac{g^{jl}}{\sqrt{g_{(kk)}}}\sqrt{g_{(ii)}}\sqrt{g_{(jj)}}\Gamma^i_{lk}v^{*k} \right]. \end{aligned} \tag{29}$$

By inserting the metric tensors and Christoffel symbols from Appendices A and B, the components

$$\frac{T^{*11}}{\rho K} = -\frac{2}{r\cos\phi}\left(\frac{\partial v^{*1}}{\partial\lambda} + \frac{J_\lambda}{\sqrt{G}}\frac{\partial v^{*1}}{\partial\zeta}\right) + \frac{2\tan\phi}{r}v^{*2} - \frac{2}{r}v^{*3}, \tag{30}$$

$$\frac{T^{*12}}{\rho K} = -\frac{1}{r\cos\phi}\left(\frac{\partial v^{*2}}{\partial\lambda} + \frac{J_\lambda}{\sqrt{G}}\frac{\partial v^{*2}}{\partial\zeta}\right) - \frac{1}{r}\left(\frac{\partial v^{*1}}{\partial\phi} + \frac{J_\phi}{\sqrt{G}}\frac{\partial v^{*1}}{\partial\zeta}\right) - \frac{\tan\phi}{r}v^{*1}, \tag{31}$$

$$\frac{T^{*13}}{\rho K} = -\frac{1}{r\cos\phi}\left(\frac{\partial v^{*3}}{\partial\lambda} + \frac{J_\lambda}{\sqrt{G}}\frac{\partial v^{*3}}{\partial\zeta}\right) + \frac{1}{\sqrt{G}}\frac{\partial v^{*1}}{\partial\zeta} + \frac{1}{r}v^{*1}, \tag{32}$$

$$\frac{T^{*22}}{\rho K} = -\frac{2}{r}\left(\frac{\partial v^{*2}}{\partial\phi} + \frac{J_\phi}{\sqrt{G}}\frac{\partial v^{*2}}{\partial\zeta}\right) - \frac{2}{r}v^{*3}, \tag{33}$$

$$\frac{T^{*23}}{\rho K} = -\frac{1}{r}\left(\frac{\partial v^{*3}}{\partial\phi} + \frac{J_\phi}{\sqrt{G}}\frac{\partial v^{*3}}{\partial\zeta}\right) + \frac{1}{\sqrt{G}}\frac{\partial v^{*2}}{\partial\zeta} + \frac{1}{r}v^{*2}, \tag{34}$$

$$\frac{T^{*33}}{\rho K} = \frac{2}{\sqrt{G}}\frac{\partial v^{*3}}{\partial\zeta} \tag{35}$$

follow. Here, the last terms in the right hand sides of each of the equations (30), (32), (33) and (35) are not contained in Doms, et.al., 2005. Analogous to this, the (much more simple) derivation for the scalar fluxes follows

$$H^i = -\rho K_s g^{ij}\nabla_j s, \tag{36}$$

from which the physical components can be derived

$$H^{*i} = -\rho K_s \sqrt{g_{(ii)}} g^{ij}\frac{\partial x'^m}{\partial x^j}\frac{\partial s}{\partial x'^m}, \tag{37}$$

or for the single components

$$H^{*1} = -\rho K_s \frac{1}{r\cos\phi}\left(\frac{\partial s}{\partial\lambda} + \frac{J_\lambda}{\sqrt{G}}\frac{\partial s}{\partial\zeta}\right), \tag{38}$$

$$H^{*2} = -\rho K_s \frac{1}{r}\left(\frac{\partial s}{\partial\phi} + \frac{J_\phi}{\sqrt{G}}\frac{\partial s}{\partial\zeta}\right), \tag{39}$$

$$H^{*3} = +\rho K_s \frac{1}{\sqrt{G}}\frac{\partial s}{\partial\zeta}, \tag{40}$$

which agrees to Doms, et.al., 2005.

Again, we want to estimate the relevance of the terms. In principle LMK shall resolve the bigger parts of deep convection. Therefore all the velocity components can have the same order and we can roughly estimate

$$\max \frac{|v^{*i}|}{r} \sim \frac{100\, m/s}{6 \cdot 10^6\, m} \sim 10^{-5} \frac{1}{s} \tag{41}$$

Compared to this any vertical deformations are much bigger, for example

$$\frac{1}{\sqrt{G}} \frac{\partial v^i}{\partial \zeta} \sim \frac{\Delta v}{\Delta z} \sim \frac{1\, m/s}{100\, m} \sim 10^{-2} \frac{1}{s}, \tag{42}$$

(we used an arbitrary value of $\Delta v \sim 1$ m/s, which is certainly not very much, because we assumed a strong wind case). Even horizontally we would expect

$$\frac{\Delta v}{\Delta x} \sim \frac{0.3\, m/s}{2800\, m} \sim 10^{-4} \frac{1}{s}. \tag{43}$$

Again, all the terms connected with the earth curvature are at least one order less than the gradient terms and therefore can be neglected. But, as before, all the terms connected with the terrain slope cannot be neglected.

4 Conclusions

Turbulent fluxes and flux divergences of momentum and arbitrary scalars were derived for the special LM/LMK coordinate system und base vectors. It occurred, that there are discrepancies between this derivation and the original LM documentation. A test of correctness of this derivation is the fact, that some of the 'new' terms can be motivated illustratively.

Nevertheless, the discrepancies are not of practical importance, because they occur only in terms corresponding to the earth curvature. The neglection of terms connected with the earth curvature can be motivated also by the following qualitative consideration. The maximum vertical length scale of turbulence is on the one hand determined by the depth of the planetary boundary layer ($L_v \sim 1$ km) and on the other hand is sureley not greater than the depth of the troposphere ($L_v \sim 10$ km), if one regards deep convection as a kind of turbulence, too. The horizontal length scale of turbulence cannot exceed very much this vertical length scale. Therefore, we have a maximum turbulence length scale of the order $L_{turb} < 10$ km. This is almost 3 orders less than the earth radius $R \sim 6400$ km. Therefore all the terms connected with the pure earth curvature can be neglected.

In summary, in the flux divergences of momentum (8)-(10) and arbitrary scalars (13) all the terms which contain the fluxes themself and not their spatial derivatives can be neglected (that means, all the second lines can be neglected). Similarly, in the turbulent fluxes of momentum (30)-(35) all the terms which contain the velocities itself and not their spatial derivatives can be neglected. In contrast, terms connected with the terrain slope have to be maintained.

Appendix A: Spherical coordinates

We use a spherical coordinate system with coordinates

$$x^1 = \lambda, \quad x^2 = \phi, \quad x^3 = r. \tag{44}$$

If one compares this with the literature, one should obey a possibly different relation with the cartesian coordinates. Here we use

$$x = r\cos\phi\cos\lambda, \quad y = r\cos\phi\sin\lambda, \quad z = r\sin\phi. \tag{45}$$

Note, that we define $\phi \in [-\pi/2, \pi/2]$ and therefore $\cos\phi \geq 0$.

For completeness we list the metric tensor

$$g_{ij} = \begin{pmatrix} r^2\cos^2\phi & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Leftrightarrow \quad g^{ij} = \begin{pmatrix} \frac{1}{r^2\cos^2\phi} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{46}$$

and the Christoffel symbols

$$\Gamma^1_{ij} = \begin{pmatrix} 0 & -\tan\phi & 1/r \\ -\tan\phi & 0 & 0 \\ 1/r & 0 & 0 \end{pmatrix},$$
$$\Gamma^2_{ij} = \begin{pmatrix} \cos\phi\cdot\sin\phi & 0 & 0 \\ 0 & 0 & 1/r \\ 0 & 1/r & 0 \end{pmatrix}, \quad \Gamma^3_{ij} = \begin{pmatrix} -r\cos^2\phi & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

Appendix B: Terrain-following coordinates

The transition to terrain-following coordinates is described by

$$x^1 = x'^1 = \lambda, \qquad x^2 = x'^2 = \phi, \qquad x^3 = r(\lambda, \phi, \zeta). \tag{48}$$

Therefore we get for the required derivatives

$$\begin{aligned} &\frac{\partial x'^1}{\partial x^1} = 1, \quad \frac{\partial x'^1}{\partial x^2} = 0, \quad \frac{\partial x'^1}{\partial x^3} = 0, \\ &\frac{\partial x'^2}{\partial x^1} = 0, \quad \frac{\partial x'^2}{\partial x^2} = 1, \quad \frac{\partial x'^2}{\partial x^3} = 0, \\ &\frac{\partial x'^3}{\partial x^1} = \frac{\partial \zeta}{\partial \lambda} \equiv \frac{J_\lambda}{\sqrt{G}}, \quad \frac{\partial x'^3}{\partial x^2} = \frac{\partial \zeta}{\partial \phi} \equiv \frac{J_\phi}{\sqrt{G}}, \quad \frac{\partial x'^3}{\partial x^3} = \frac{\partial \zeta}{\partial z} \equiv -\frac{1}{\sqrt{G}}, \end{aligned} \tag{49}$$

where we introduced the abbreviations J_λ, J_ϕ and $\sqrt{G}$ analogous to the LM-documentation (Doms and Schättler, 2002).

References

Doms, G. and U. Schättler, 2002: A Description of the Nonhydrostatic Regional Model LM, Part I: Dynamics and Numerics. Deutscher Wetterdienst, Offenbach, Germany, November 2002.

Doms, G., J. Förstner, E. Heise, H.-J. Herzog, M. Raschendorfer, T. Reinhardt, and G. Vogel, 2005: A Description of the Nonhydrostatic Regional Model LM, Part II: Physical Parameterizations. Deutscher Wetterdienst, Offenbach, Germany, April 2005.

Förstner J., H.-J. Herzog, and G. Vogel, 2005: Implementation of a 3D-turbulence parameterization for the very short range forecast model LMK. *WGNE Blue Book*, 2005.
(see: `http://www.cmc.ec.gc.ca/rpn/wgne/`).

Herzog H.-J., U. Schubert, G. Vogel, A. Fiedler, and R. Kirchner, 2002a: LLM - the High-Resolving Nonhydrostatic Simulation Model in the DWD-Project LITFASS (Part I: Modelling Technique and Simulation Method) COSMO Technical Report No. 4. Deutscher Wetterdienst, Offenbach, Germany, March 2002.

Herzog H.-J., G. Vogel, and U. Schubert, 2002b: LLM - a nonhydrostatic model applied to high-resolving simulations of turbulent fluxes over heterogeneous terrain. *Theor. Appl. Climatol.*, 73, 67–86, 2002.

Raschendorfer M., 2004: A new TKE-based Scheme for Vertical Diffusion and Surface-Layer Transfer. Deutscher Wetterdienst, Offenbach, Germany, 2004.

Stephani H., 1988: *Allgemeine Relativitätstheorie.* Dt. Verlag d. Wiss., Berlin, 1988.