

High Order Flux based Conservation Schemes

J. Steppeler, Langen 2009

The definition of high (fourth) order
finite difference schemes resulting
from low order approximations of
conservation laws

Icosahedral meshes are irregular to a degree

- The common procedure is to regularise them such that the finite volume method becomes second order.
- Point flux methods allow to define modified fluxes at cell interfaces such that the finite volume approximation becomes fourth order even on irregular grids.
- Therefore fourth order methods on the original icosahedral grid will be possible, even for cut cells.

The challenge of global cloud resolving weather and climate modelling

- Nonhydrostatic
- High accuracy/ high order of approximation (3 or 4)
- Conservation
- Z-coordinate
- Efficiency
- **Show that the new dynamics gives a boost in wea and clim performance.**

High and low order mass formula

Spectral, spectral elements, finite elements:

$$mass_element = \int_{dF} spectral_rep_of_density \circ dx$$

Finite Difference:

$$mass_element = \int_{dF} \rho dx \approx \rho_i dx$$

Question: is FD conservation possible with a high order approximation of the density equation?

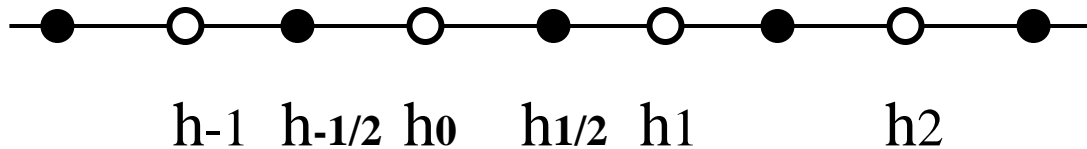
Existing models:

- **COSMO**: order 3+, **no conservation of mass**
- **WRF**: Order 3 to 5 in many terms, but order 2 in density approximation, overall order 2, **conservation of mass**
- Currently existing FD models are either conserving or high order. **NOAA** is researching into a model having both properties.

Definition of flux based conservation schemes

$$\frac{\partial h}{\partial t} = \frac{\partial uh}{\partial x}; f = uh; u = 1$$

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial x}; f = h$$



Finite difference flux based conservation:

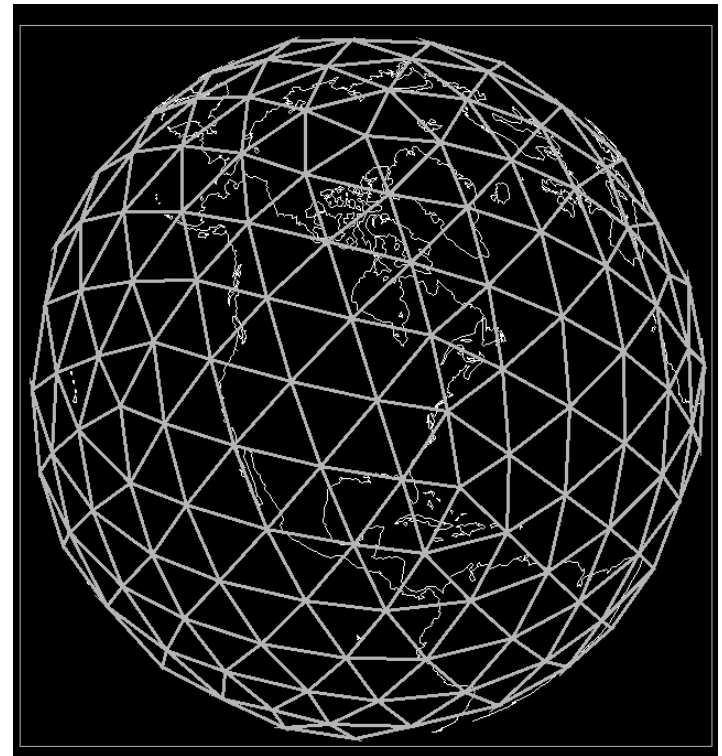
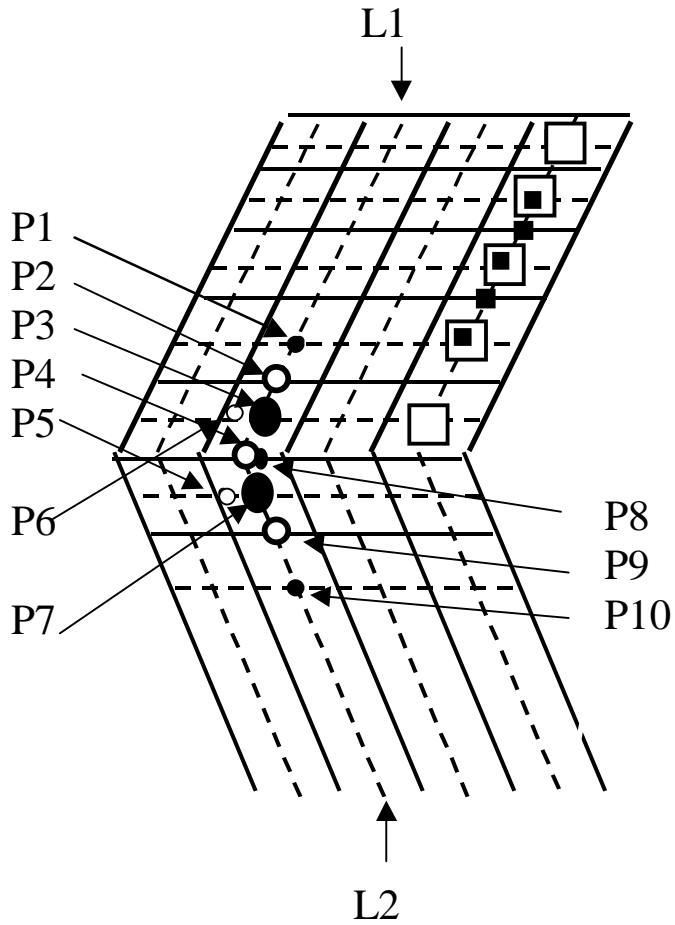
$$dx \frac{\partial h_0}{\partial t} = h_{1/2} - h_{-1/2}; \Rightarrow \frac{\partial H}{\partial t} = \sum_i dx \frac{\partial h_i}{\partial t} = 0$$

Properties of flux based conservation schemes

- If the $h_{1/2}$ are exactly computed by spectral method, the flux based scheme is of order 2 at most
- If the $h_{1/2}$ are computed by fourth order interpolation from the h_i , the flux based scheme is of order 2 at most
- **If $h_{1/2}$ deviate from the exact value by more than order 2, it is possible to define them in such a way to result into a 3rd or 4th order approximation of the density equation. Such coarsely approximated fluxes are called point fluxes.**

Grids with fault lines

Fault lines limit areas of changing grid orientation or resolution

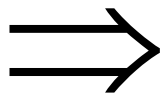
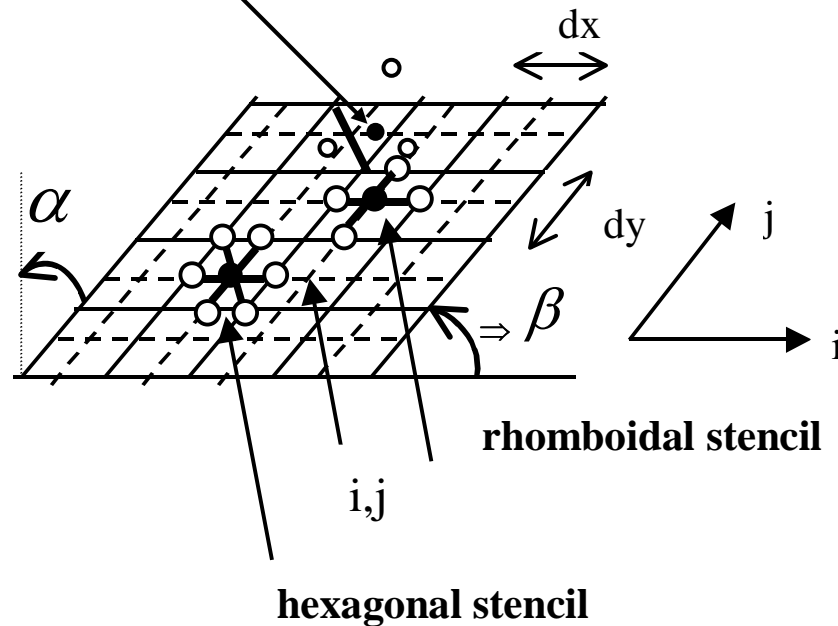


Grids with fault lines

- At grid fault lines linear interpolation and differencing leads to natural conservation
- Higher order interpolation needs to be specifically designed to achieve conservation. **“conserving interpolation”**
- FV schemes work without interpolation, but may need some interpolation to define the flux points. The Japanese scheme obtains sufficient accuracy only if the grid is “Regularised”
- The interpolation to obtain the fluxes in FV schemes is often high order (Vandermann scheme at NOAA)
- The method suggested here defines fluxes in FV schemes in low order such that the conservation equation is approximated in high order

Hexagonal and triangular stencils and elements in the icosahedral grid

Triangular stencil



The triangular and the hexagonal grids and systems of control volumes are inherent in the icosahedral grid

How to define point fluxes $f^*_{-1/2}, f^*_{1/2}$

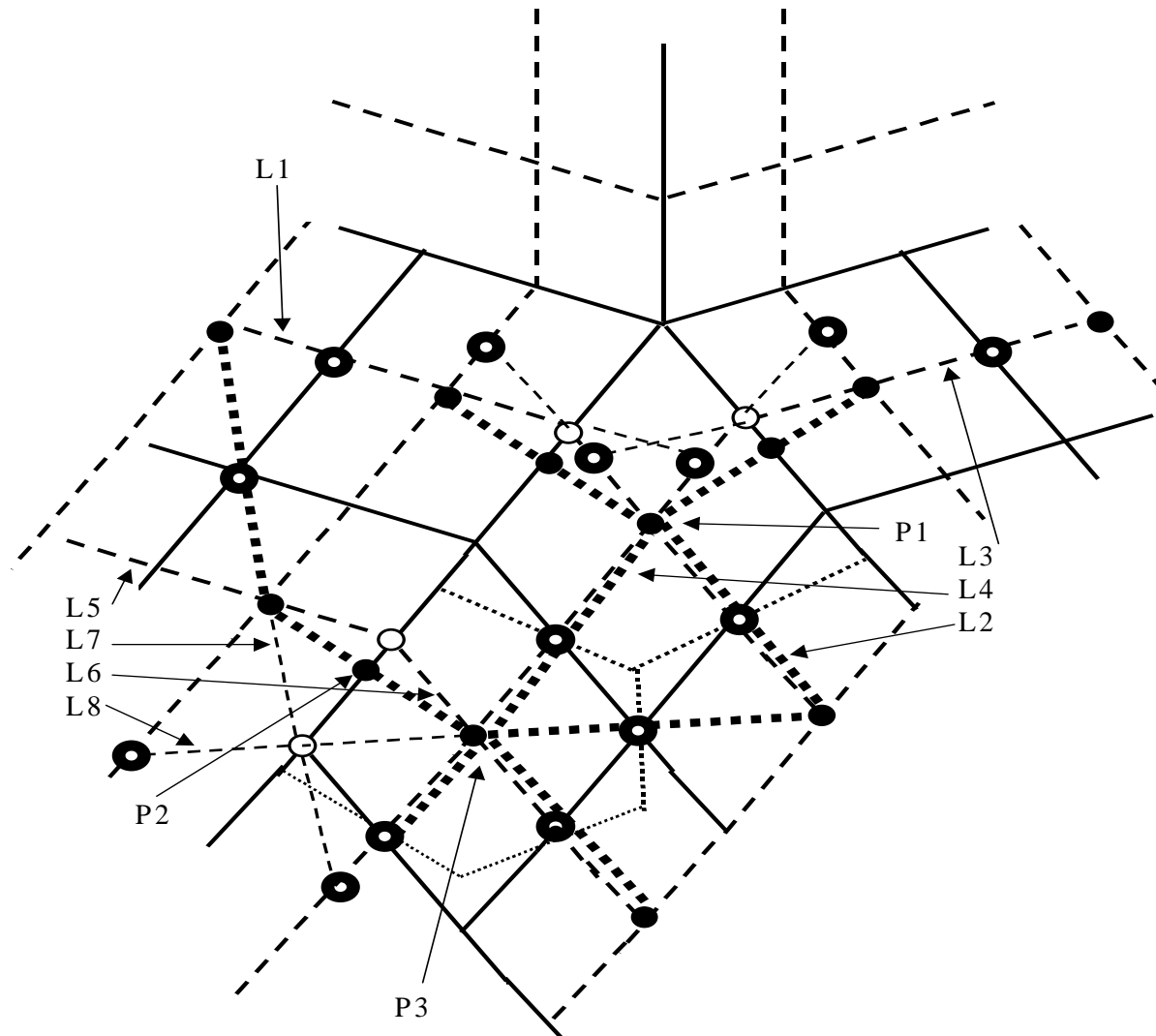
Stencil: $i \in \{-2, -1, 0, 1, 2\}$

Point flux definition: $f^*_{-1/2} = \sum_i a_i f_i; f^*_{1/2} = \sum_i b_i f_i$

Equation for the a_i, b_i : $(f^*_{1/2} - f^*_{-1/2}) / dx - \frac{\partial f}{\partial x} = o(dx^5)$

for all polynomial functions $f(x) = x^n, n \in \{0, 1, 2, 3, \dots\}$

Flux compensation scheme for grid near the pole



Possible applications

- High order and conserving methods on the original great circle icosahedron using only 1-d interpolation
- Application to (4th) order cut cells and changing resolution

Conclusions

- FD or FV flux balance equations are of second order approximation, when the true surface fluxes are approximated in third or higher order
- Point fluxes are low order approximations of the true flux at a point. They can be determined in such a way that the conservation equation is approximated in fourth order
- some complications arise for the icosahedron
- Real life models with conservation, high order and z-coordinates are not yet on the market, but under investigation at several places.** Do they make a difference in performance?