An Improved Third Order Vertical Advection Scheme for the Runge-Kutta Dynamical Core

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Motivation:

in a convection-permitting model (like COSMO-DE) the vertical advection plays a much bigger role than in a convection-parameterising model → try to achieve higher accuracy in the vertical advection of dynamic variables \((u,v,w,T',p')\), too

COSMO-model up to now: vertically implicit centered diff. 2nd order

WRF: vertically explicit upwind scheme (3rd order)

• advantages:
  • Fits best to the explicit horizontal advection and the Runge-Kutta-scheme
  • Relatively easy to implement

• disadvantages:
  • Limitation of Courant number: \(C_x + C_y + C_z < 1.4\) (Baldauf, 2008, JCP) →
    • WRF uses smaller time steps (~15 sec for dx=3km)
    • WRF uses a vertical ‘velocity brake’

→ Keep the vertically implicit scheme, but try a higher order of approximation
(COSMO priority project 'Runge-Kutta', Task 8)
Several implicit advection schemes (2-timelevels)

\[
\frac{\phi^{n+1} - \phi^n}{\Delta t} = \beta A_z(\phi^{n+1}) + (1 - \beta) A_z(\phi^n)
\]

To achieve higher order in time, too, we will use \(\beta=1/2\) (‘pure’ Crank-Nicholson)

\(A_z\) denotes a spatial discretisation of the vertical advection operator:
- Centered differences 2nd order (spatially and temporally) (‘CN2’) current
- Upwind 3rd order (spatially) (‘CN3’) new
- Upwind 3rd order (spatially and temporally) (‘CN3Crow’) new
  derivation analogous to the explicit Crowley-schemes
  (Tremback et al., 1987, MWR)
Pure advection test:
Only implicit scheme (i.e. $u=0$), $w=2$, $dt=0.25$ → $C_z=0.5$
Pure advection test:
Only implicit scheme (i.e. u=0), w=2, $dt=1.5 \rightarrow C_z=3$
The current 3-stage RK-scheme in the COSMO-model

Wicker, Skamarock (2002) MWR

solve the implicit scheme: \[
\frac{\tilde{\phi} - \phi^n}{\Delta t/3} = \beta A_z(\tilde{\phi}) + (1 - \beta)A_z(\phi^n) + A_x(\phi^n) + P(\phi^n)
\] (1)

... and define its tendency: \[L(\phi^n) := \frac{\tilde{\phi} - \phi^n}{\Delta t/3}\] (2)

1. RK-substep: \[\phi^* = \phi^n + \frac{\Delta t}{3}L(\phi^n)\] (3)

fast waves with tendency \[\frac{\phi^* - \phi^n}{\Delta t/3}, \text{ starting at } \phi^n \Rightarrow \phi^*\] (4)

solve: \[
\frac{\tilde{\phi} - [\alpha\phi^n + (1-\alpha)\phi^*]}{\Delta t/2} = \beta A_z(\tilde{\phi}) + (1 - \beta)A_z(\phi^*) + A_x(\phi^*) + P(\phi^n)
\] (5)

... and define its tendency: \[L(\phi^*) := \text{ lhs. of the above expression}\] (6)

2. RK-substep: \[\phi^{**} = \phi^n + \frac{\Delta t}{2}L(\phi^*)\] (7)

fast waves with tendency \[\frac{\phi^{**} - \phi^n}{\Delta t/2}, \text{ starting at } \phi^n \Rightarrow \phi^{**}\] (8)

solve: \[
\frac{\tilde{\phi} - [\alpha\phi^n + (1-\alpha)\phi^{**}]}{\Delta t} = \beta A_z(\tilde{\phi}) + (1 - \beta)A_z(\phi^{**}) + A_x(\phi^{**}) + P(\phi^n)
\] (9)

... and define its tendency: \[L(\phi^{**}) := \text{ lhs. of the above expression}\] (10)

3. RK-substep: \[\phi^{n+1} = \phi^n + \Delta t L(\phi^{**})\] (11)

fast waves with tendency \[\frac{\phi^{n+1} - \phi^n}{\Delta t}, \text{ starting at } \phi^n \Rightarrow \phi^{n+1}\] (12)

In the following: \[A_x = \text{ upwind 5th order}\]
The current 3-stage RK-scheme in the COSMO-model

CN2, $C_z=2.4$

CN2, $C_z=2.0$

CN2, $C_z=1.6$

CN2, $C_z=1.0$

CN2, $C_z=0.5$

$C_x = 0.5 C_z$

Test 20081022_a
• The current scheme shows a certain damping, which seems to be typically for implicit schemes inside of a Runge-Kutta-scheme
• The vertical advection is not unconditionally stable, despite the fact, that an implicit scheme is used (but up to now this was not a problem in all COSMO-DE or –EU runs)

Remark: \( \alpha = 0 \) (i.e. without an overdamping):
• Bigger dispersion error
• Instability sets in much earlier (at \( dt=0.8 \rightarrow C_z=1.6 \))
New proposal: complete operator splitting

\[ \frac{\tilde{\phi} - \phi^n}{\Delta t} = R_x(\phi^n) \quad R_x = \text{complete RK3-scheme, but without } A_z \]

\[ \frac{\phi^{n+1} - \tilde{\phi}}{\Delta t} = \beta A_z(\phi^{n+1}) + (1 - \beta) A_z(\tilde{\phi}) \]

advantages:
- no implicit scheme occurs inside of the RK
- if the numerical operators commute, then the stability properties of the single operators are passed on to the whole scheme
  \textit{(LeVeque and Oliger, 1983)}
- 'overdamping' is not needed (\( \rightarrow \alpha=0 \) is possible)
- 'expensive' vertical advection is called only once / timestep
Complete operator splitting, $dt=0.25 \rightarrow C_z=0.5$
Complete operator splitting, \( \text{dt}=1.2 \rightarrow C_z=2.4 \)

- CN2
- CN3
- CN3Crow
Question: does the 'complete operator splitting' work together with fast (sound-, gravity-) waves? (splitting-error, stability, ...)

Linearised shallow water equations

\[ \frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + V_0 \frac{\partial u}{\partial y} = -g \frac{\partial h}{\partial x} + M_x \]
\[ \frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} + V_0 \frac{\partial v}{\partial y} = -g \frac{\partial h}{\partial y} + M_y \]
\[ \frac{\partial h}{\partial t} + U_0 \frac{\partial h}{\partial x} + V_0 \frac{\partial h}{\partial y} = -H_0 \frac{\partial h}{\partial y} \]

with \( D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \)

Advective transport with velocity \((U_0, V_0)\)

fast wave expansion with velocity \( c_{wav} = \sqrt{gH_0} \)

Treat \text{y-adv.} as ‘vertical’ advection

Divergence damping:

\[ M_x = \alpha_{Div} \frac{\partial D}{\partial x} \]
\[ M_y = \alpha_{Div} \frac{\partial D}{\partial y} \]
For reference:
upwind5 in y-direction

New scheme:
CN3Crow in y-direction

Height field

dt=0.6

dt=0.25

dt=0.25

dt=0.6

dt=1
Do better combinations exist between horizontal and vertical advection?

- complete operator splitting, but with horizontal advection as 'rhs':
  \[
  \frac{\phi^{n+1} - \phi^n}{\Delta t} = \beta A_z(\phi^{n+1}) + (1 - \beta) A_z(\phi^n) + R_x(\phi^n)
  \]
  → adv. test unstable!
  It seems generally not to be a good idea to use explicit parts as a rhs of an implicit scheme
- tendencies of the vertical advection analogous to the physical tendencies
  → adv. test unstable
- implicit scheme only in 3rd RK-sub step → adv. test unstable
- mixing: explicit in RK 1st+2nd sub step, implicit in 3rd RK-sub step
  → adv. test: strongly damping or unstable
- 'partial operator splitting': → adv. tests ok., but shallow water equations unstable

\[
\begin{align*}
\phi^* &= \phi^n + \frac{\Delta t}{3} A_x(\phi^n) \\
\phi^{**} &= \phi^n + \frac{\Delta t}{2} A_x(\phi^*) \\
\phi^{***} &= \phi^n + \Delta t A_x(\phi^{**}) \\
\phi^{n+1} &= \phi^{***} + \Delta t \left[ \beta A_z(\phi^{n+1}) + (1 - \beta) A_z(\phi^{***}) \right]
\end{align*}
\]
Real case study:
COSMO-DE (2.8 km resolution) for the '01.08.2008', 0 UTC run
1h-precipitation sum at 13 UTC

'New VA' = 'Complete operator splitting' with 'CN3Crow'
Real case study:
COSMO-DE (2.8 km resolution) for the '01.08.2008', 0 UTC run
1h-precipitation sum at 14 UTC
Real case study:
COSMO-DE (2.8 km resolution) for the '01.08.2008', 0 UTC run
1h-precipitation sum at 15 UTC
Real case study: COSMO-DE (2.8 km resolution) for the '01.08.2008', 0 UTC run
1h-precipitation sum at 16 UTC
01.08.2008, 0 UTC
21 h precipitation sum
Diff. 'New – Old. VA'
01.08.2008, 0 UTC, mean/max values

\[ \langle \frac{d p_{\text{surf}}}{dt} \rangle \]

\[ \langle E_{\text{kin}} \rangle \]

max \( v_h \)

max \( w \)
Efficiency:

CN2 / current scheme:
solves a tridiagonal system: comp. effort ~ 3 N
call in every RK-substep = 3 times / timestep

CN3, ... / complete operator splitting:
solves a pentadiagonal system: comp. effort ~ 13 N
call 1 times / timestep

uses 'Numerical recipes' routines `bandec`, `banbks`, which where
optimized for vector computers (NEC SX-8R)
Summary

The current implicit vertical advection scheme possess a relatively strong damping and is formally not unconditionally stable.

From all of the tested alternatives only the 'complete operator splitting' (= vertical advection outside of the RK-scheme) with CN3 or CN3Crow has proven to be superior:
- improved advection properties in idealized advection tests
- unconditionally stable in $C_z$
- works also in combination with fast waves
- plausible results in idealized and real cases
- runs stable for several COSMO-DE (2.8 km) simulations (summer period);
  but L. Torrisi (CNMCA, Rome) found an unstable case with a 7 km resolution!
- computational amount is only slightly increased

Outlook
- inspection of an unstable case; winter time cases
- Synoptic verification of a longer COSMO-DE period (August 2008)
Demonstrate the ability of the implicit schemes: 2D-advection tests

In the following:

- the 3-stage Runge-Kutta is used as time integration scheme
- the horizontal advection always is a 5th order upwind scheme

Wicker, Skamarock, 2002, MWR
Test 'solid body rotation' (1 turn around)
COSMO-model:
behaviour with complete dynamical core (RK3) + simplified physics (only cond./evap
Weisman, Klemp (1982)-test case
$q_c$, after 30 min.

CN2 /alt

CN3 / kompl. Oper-splitt.
gprof: CN2 / bisheriges Schema

ngranularity: Each sample hit covers 4 bytes. Time: 10324.90 seconds

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### gprof: CN3Crow / komplettes Operatorsplitting

*ngranularity: Each sample hit covers 4 bytes. Time: 10248.44 seconds*

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01.08.2008, 0 UTC, mean/max values

max $v_h$

max $w$

Realer Testfall ('22.08.2008, 0 UTC')

mittlere Drucktendenz:

neues Schema u.U. etwas weniger gestört

max $v_h$

max $w$