Discontinuous Galerkin Method

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Basis Function Choices for EBG Methods

• Modal Space
  – Spectral (amplitude-frequency) space
  – Orthogonal functions (e.g., Legendre, PKD)

• Nodal Space
  – Physical variable space
  – Cardinal functions (e.g., Lagrange polynomials)

• Continuous
  – FE and SE Methods (local)
  – Spectral Transform Method (global)

• Discontinuous
  – FV and DG Methods (local)
Properties of Continuous EBG Methods

- High order like spherical harmonics yet local like finite element methods
  - globally conservative = good for hydrostatic primitive equation
- Theoretically optimal for self-adjoint operators (elliptic equations)
  - Excellent for incompressible Navier-Stokes
  - also extremely good for non-self-adjoint operators (hyperbolic equations)
- Simple to construct efficient semi-implicit time-integrators
  - Semi-implicit = nonlinear terms are explicit and linear terms are implicit
Properties of Discontinuous EBG Methods

- High order like Continuous Galerkin
- Completely local in nature
  - no global assembly/DSS required as in CG (truly local)
- High order generalization of the FV
  - locally and globally conservative = excellent choice for non-hydrostatic equations (mesoscale models)
  - upwinding implemented naturally (via Riemann solvers)
- Designed for hyperbolic equations (i.e., shock waves)
  - Simple provision for elliptic equations
- Not so straightforward to construct efficient semi-implicit time-integrators (inherently nonlinear)
Discretization by Continuous EBG Methods

• Primitive Equations:
  \[ \frac{\partial \bar{q}}{\partial t} = S(\bar{q}) \]

• Approximate the solution as:
  \[ \bar{q}_N(x, y, z) = \sum_{i=1}^{M_N} \psi_i(x, y, z) \bar{q}_i \]

• Write Primitive Equations as:
  \[ R(\bar{q}_N) \equiv \frac{\partial \bar{q}_N}{\partial t} - S(\bar{q}_N) = 0 \]

• Weak Problem Statement: Find
  \[ \bar{q}_N \in H^1(\Omega) \forall \psi \in H^1 \]
  – (which assumes \( C^0 \))
  – such that
  \[ \int_{\Omega} \psi R(\bar{q}_N) d\Omega = 0 \]
  – where
  \[ \Omega = \bigcup_{e=1}^{N_e} \Omega_e \]
Discretization by Discontinuous EBG Methods

- Primitive Equations: \( \frac{\partial \bar{q}}{\partial t} = S(\bar{q}) \)

- Approximate the solution as: \( \bar{q}_N(x, y, z) = \sum_{i=1}^{M_N} \psi_i(x, y, z) \bar{q}_i \)

- Write Primitive Equations as: \( R(\bar{q}_N) \equiv \frac{\partial \bar{q}_N}{\partial t} - S(\bar{q}_N) = 0 \)

- Weak Problem Statement: Find \( \bar{q}_N \in L^2(\Omega) \forall \psi \in L^2 \) (which does not assume \( C^0 \))
  - such that \( \int_{\Omega} \psi R(\bar{q}_N) d\Omega = 0 \)
  - where \( \Omega = \bigcup_{e=1}^{N_e} \Omega_e \)
Difference between CG and DG

- Conservation Law: \( \frac{\partial q}{\partial t} + \nabla \cdot \overline{F} = 0 \)

- Weak Form: \( \int_\Omega \psi \left( \frac{\partial q}{\partial t} + \nabla \cdot \overline{F} \right) d\Omega = 0 \)

- Approximation: \( q_N(\xi, \eta) = \sum_{j=1}^{M_N} \psi_j(\xi, \eta) q_j \)

  \( F_N(\xi, \eta) = \sum_{j=1}^{M_N} \psi_j(\xi, \eta) F_j \)

- Integration By Parts:

\[
\int_\Omega \left( \psi \frac{\partial q_N}{\partial t} + \nabla \cdot (\psi \overline{F_N}) - \nabla \psi \cdot \overline{F_N} \right) d\Omega = 0
\]
Difference between CG and DG

• Using Divergence Theorem and $C^0$ continuity of CG:

\[
\int_{\Omega} \left( \psi \frac{\partial q_N}{\partial t} - \nabla \psi \cdot \overline{F}_N \right) d\Omega = -\int_{\Gamma} \psi \overline{n} \cdot \overline{F}_N^* \ d\Gamma
\]

• Using Divergence Theorem (DG):

\[
\int_{\Omega} \left( \psi \frac{\partial q_N}{\partial t} - \nabla \psi \cdot \overline{F}_N \right) d\Omega = -\int_{\Gamma} \psi \overline{n} \cdot \overline{F}_N^* \ d\Gamma
\]

• Constant $\psi$ for DG yields FV:

\[
\int_{\Omega} \frac{\partial q}{\partial t} \ d\Omega = -\int_{\Gamma} \overline{n} \cdot \overline{F}^* \ d\Gamma
\]
Continuous EBG Methods

(Matrix Form)

Global Matrix Form

$$M_{I,J} \frac{\partial q_J}{\partial t} + \left( \overline{D_{I,J}} \right)^T \overline{F_J} = 0$$

Element-wise Approximation

$$q_{N}^{(e)} = \sum_{j=0}^{N} \psi_j q_j^{(e)} \quad \overline{F_N}^{(e)} = \sum_{j=0}^{N} \psi_j \overline{F_j}^{(e)}$$

Element Matrices

$$M_{i,j}^{(e)} = \int_{\Omega_e} \psi_i \psi_j d\Omega_e \quad \overline{D_{i,j}}^{(e)} = \int_{\Omega_e} \psi_i \nabla \psi_j d\Omega_e$$

DSS

$$M_{I,J} = \sum_{e=1}^{N_E} M_{i,j}^{(e)} \quad \overline{D_{I,J}} = \sum_{e=1}^{N_E} \overline{D_{i,j}}^{(e)}$$

Domain Decomposition

$$\Omega = \bigcup_{e=1}^{N_E} \Omega_e$$
Discontinuous EBG Methods

(Matrix Form)

Local Matrix Form

\[ M_{i,j}^{(e)} \frac{\partial q_j^{(e)}}{\partial t} + \left( \overline{D_{i,j}}^{(e)} \right)^T \overline{F}_j^{(e)} = \left( \overline{M_{i,j}}^{(e)} \right)^T \left( \overline{F}_j^{(e)} - \overline{F}_j^{(*)} \right) \]

Element-wise Approximation

\[ q_N^{(e)} = \sum_{j=0}^{N} \psi_j q_j^{(e)} \quad \overline{F}_N^{(e)} = \sum_{j=0}^{N} \psi_j \overline{F}_j^{(e)} \]

Element Matrices

\[ M_{i,j}^{(e)} = \int_{\Omega_e} \psi_i \psi_j d\Omega_e \quad \overline{D}_{i,j}^{(e)} = \int_{\Omega_e} \psi_i \nabla \psi_j d\Omega_e \]

\[ \overline{M}_{i,j}^{(e)} = \int_{\Gamma_e} \psi_i \psi_j n d\Gamma_e \]

Domain Decomposition

\[ \Omega = \bigcup_{e=1}^{N_E} \Omega_e \]
Continuous vs. Discontinuous EBG Methods
(Stencil)
Continuous EBG Methods
(Global Grid Points due to $C^0$)

\[
\int_{\Omega} \left( \psi \frac{\partial q_N}{\partial t} - \nabla \psi \cdot F_N \right) d\Omega = -\int_{\Gamma} \psi \overline{n} \cdot F_N^* d\Gamma
\]
Discontinuous EBG Methods
(Local Grid Points)

\[
\int_{\Omega} \left( \psi \frac{\partial q_N}{\partial t} - \nabla \psi \cdot \overrightarrow{F_N} \right) d\Omega = -\int_{\Gamma} \psi \overrightarrow{n} \cdot \overrightarrow{F_N}^* d\Gamma
\]
Difference between CG and DG

\[ \int_{\Omega} \left( \psi \frac{\partial q_N}{\partial t} - \nabla \psi \cdot \overline{F_N} \right) \, d\Omega = -\int_{\Gamma} \psi \bar{n} \cdot \overline{F_N}^* \, d\Gamma \]
Advantage of the DG Method

The Power of DG is due to the Flux Integral

\[
\int_{\Omega} \left( \psi \left( \frac{\partial q_N}{\partial t} - \nabla \psi \cdot \bar{F}_N - \psi S(q_N) \right) \right) d\Omega = -\int_{\Gamma} \psi \bar{n} \cdot \bar{F}_N^* d\Gamma
\]
Advantage of the DG Method

Flux Integral is the Key for 2 Reasons:

– It is the only means by which elements communicate (truly local)
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DG Communication Stencil

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The DG communication stencil of the element $T$ is given by the solid triangles (3 triangles) whereas the CG stencil of the element $T$ is given by the solid and dashed triangles (triangles).
Flux Integral is the Key for 2 Reasons:

- It is the only means by which elements communicate (truly local)
- It allows a way to introduce semi-analytic methods (Riemann Solvers)
Problem Statement

\[
\int_{\Omega} \left( \psi \frac{\partial q_N}{\partial t} - \nabla \psi \cdot \bar{F}_N - \psi S(q_N) \right) d\Omega = \int_{\Gamma} \psi \, \bar{n} \cdot \bar{F}_N^* \, d\Gamma
\]

Name of the Game is to approximate:

\[
\bar{n} \cdot \bar{F}_N^* = R^{-1} \hat{F}^* (Rq_N)
\]

Where \( R \) (for Rotational Invariant PDEs) maps:

\[
q = (\phi, \phi u, \phi v, \phi w)^T \quad \rightarrow \quad \hat{q} = (\phi, \phi u_n, \phi u_t, \phi u_r)^T
\]
Advantage of the DG Method
(Numerical Flux and Riemann Problem)

- Using the 1D Rotation Mapping we solve the Riemann Problem for:

\[ \hat{q} = (\phi, \phi u_n, \phi u_r, \phi u_r)^T \]

- The Associated Riemann Problem is:

- Where \( F^* \) is selected to best approximate the exact solution (e.g., Godunov, Osher, Roe, Rusanov, HLLC)
These results are for a scale contraction problem (passive advection of a discontinuous function of fluid). The mass profile along the Equator are shown for the CG and DG methods using $N=8$ polynomials. Even with strong spatial filtering, the CG method experiences Gibbs phenomena while the DG method only feels slight oscillations.
These results are for a shock wave dynamics problem (cylindrical shock wave on a stationary sphere). The solution profile along the Equator are shown for the CG (left panel) and DG (right panel) methods using \( N=1 \) polynomials. Even with strong spatial filtering, the CG method experiences Gibbs phenomena while the DG method only feels a small undershoot at the base of the shock fronts.
Applications of the DG Method

• Shallow Water Equations on the Sphere
• Mesoscale Atmospheric Model
• Coastal Ocean Model
Shallow Water Equations on the Sphere
(Governing Equations)

\[ \frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{u}) = 0 \]  
(Mass)

\[ \frac{\partial \varphi \mathbf{u}}{\partial t} + \nabla \cdot \left( \varphi \mathbf{u} \otimes \mathbf{u} + \frac{1}{2} \varphi^2 \mathbf{I}_3 \right) + \frac{2\Omega z}{a^2} (\kappa \times \varphi \mathbf{u}) = 0 \]  
(Momentum)

\[ \varphi = gh \]  
(Geopotential)

\[ \mathbf{u} = (u, v, w)^T, \]
\[ \mathbf{x} = (x, y, z)^T, \]
\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)^T \]
These results show the exponential (spectral) convergence of the SE and DG - that is, the error decreases exponentially with the order $N$ of the basis function

$$\text{error} \propto O(\Delta x^{N+1})$$
Shallow Water Equations on the Sphere
(Riemann problem)
Applications of the DG Method

• Shallow Water Equations on the Sphere
• Mesoscale Atmospheric Model
• Coastal Ocean Model
Mesoscale Atmospheric Model
(Governing Equations)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) = 0 \quad \text{(Mass)} \]

\[ \frac{\partial \rho \bar{u}}{\partial t} + \nabla \cdot \left( \rho \bar{u} \otimes \bar{u} + P \bar{I}_3 \right) = -f(\bar{k} x \rho \bar{u}) - \rho g \bar{k} \quad \text{(Momentum)} \]

\[ \frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta \bar{u}) = 0 \quad \text{(Energy)} \]

\[ P = P_0 \left( \frac{\rho R\theta}{P_0} \right)^{c_p/c_v} \quad \text{(Equation of State)} \]

\[ \theta = \frac{T}{\pi} \quad \text{(Definitions)} \]

\[ \bar{u} = (u, w)^T, \]
\[ \bar{x} = (x, z)^T, \]
\[ \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial z})^T \]
Mesoscale Atmospheric Model
(Rising Thermal Bubble)
Applications of the DG Method

- Shallow Water Equations on the Sphere
- Mesoscale Atmospheric Model
- Coastal Ocean Model
Coastal Ocean Model
(Governing Equations)

\[ \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) = 0 \]  \hspace{1cm} \text{(Mass)}

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \left( \phi \mathbf{u} \otimes \mathbf{u} + \frac{1}{2} \phi^2 \mathbf{I}_2 - \mathbf{q} \right) + \frac{2 \Omega z}{a^2} (\mathbf{k} \times \phi \mathbf{u}) = g \frac{\tau}{\rho} - \gamma \phi \mathbf{u} \]  \hspace{1cm} \text{(Momentum)}

\[ \mathbf{q} = \nu \nabla \phi \mathbf{u} \]  \hspace{1cm} \text{(Temporary Variable: Such that)}

\[ \mathbf{u} = (u, v)^T, \hspace{1cm} \mathbf{x} = (x, y)^T, \hspace{1cm} \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)^T, \hspace{1cm} \phi = gh \]  \hspace{1cm} \text{(Geopotential)}
Coastal Ocean Model
(Rossby Soliton Waves in a Channel)