The Runge-Kutta method

Advanced Numerics Seminar

Langen, 08.03.2006

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Outline

- View of LMK-Dynamics, -Numerics
- Time-splitting schemes
  - General remarks
  - Runge-Kutta + Advection
  - Sound-Advection-Buoyancy-System
  - Idealized tests
  - Implicit Vertical Advection
- Coupling Dynamics and Physics
- Dynamic Lower Boundary Condition

LMK - Dynamics

- Model equations: non-hydrostatic, full compressible, advection form
- Base state: hydrostatic
- Prognostic variables: cartesian wind components u, v, w
  pressure perturbation p‘, Temperature T (or $T'=T-T_0$)
  humidity var. q_v, q_c, q_i, q_r, q_s, q_g
  TKE
- Coordinate systems: rotated geographical coordinates
  generalized terrain-following height coordinate
  user-defined vertical stretching

Open questions:

- use shallow or deep atmospheric equations?
  i.e. are terms ~w in advection earth curvature and Coriolis force important?
  (diploma thesis) (e.g. Staniforth, Wood, 2003)
- diabatic terms in p-equation? (lecture H. Herzog)

Equation system of LM/LMK in spherical coordinates

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u - \frac{\tan\phi}{r}uv + \cancel{\frac{1}{r}uw} &= -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + 2\Omega(v\sin\phi - \cancel{w\cos\phi}) + M_u \\
\frac{\partial v}{\partial t} + \mathbf{v}\cdot\nabla v + \frac{\tan\phi}{r}u^2 + \cancel{\frac{1}{r}vw} &= -\frac{1}{\rho r}\frac{\partial p}{\partial\phi} - 2\Omega u\sin\phi + M_v \\
\frac{\partial w}{\partial t} + \mathbf{v}\cdot\nabla w - \cancel{\frac{u^2+v^2}{r}} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + \cancel{2\Omega u\cos\phi} + M_w \\
\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p &= -\frac{c_p}{c_v}p\nabla\cdot\mathbf{v} \\
\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T &= -\frac{1}{\rho c_v}p\nabla\cdot\mathbf{v} + \frac{1}{\rho c_p}(Q_{ph} + M_T - \nabla\cdot\mathbf{R}) \\
\rho\frac{\partial q_x}{\partial t} + \rho\mathbf{v}\cdot\nabla q_x &= -\nabla\cdot\mathbf{P}_x - \nabla\cdot\mathbf{F}_x - I_x, \qquad (x = v, c, i, r, s, g) \\
p &= \rho R_d(1 + (R_v/R_d - 1)q_v - q_c - q_r - q_s - q_i - q_g)T
\end{aligned}
$$

additionally:

- introduce a hydrostatic, steady base state
- Transformation to terrain-following coordinates
- shallow/deep atmosphere

Numerics

	LM	LMK
Grid	horizontal: Arakawa-C vertical: Lorenz	horizontal: Arakawa-C vertical: Lorenz
Time integration	3-timelevels: Leapfrog	2-timelevels: Runge-Kutta 2. order, 3. order, 3. order TVD
Advection u, v, w, T, p‘	horizontal: centered diff. 2. order vertical: implicit 2. order	horizontal: upwind 3., 5. order centered diff. 4., 6. order vertical: implicit 2. order implicit 3. order
Advection q_v, q_c, q_i, q_r, ..., TKE	qv, qc: centered diff.2. order qi: Lin, Rood qr, qs: Semi-Lagrange (trilin.)	Bott-2, conservation form or Semi-Lagrange, tricubic/trilin.
Smoothing	Divergence damping 4. order diffusion Asselin-filtering	Divergence damping 4. order diffusion ?

Time scales of atmospheric processes

	Process		**char. velocity, ...**	**char. time** (dx ~ 3 km or dz ~ 30 m)	**type**
Dynamics	Advection,	horiz.	0...10...100 m/s	∞...300...30 s	slow
		vert.	0...30 m/s	∞...1 s	(implicit)
	Sound,	horiz.	330 m/s	10 s	fast
		vert.	330 m/s	0.1 s	(implicit)
	Buoyancy			100 s (=1/N)	(slow)
	Gravity waves,	horiz.	0...200 m/s	∞...15 s	fast
	Coriolis			10000 s (=1/f)	slow
Physics	Diffusion,	vert.	0...10...100 m²/s	∞...100 s ... 10 s	(implicit)
	Sedimentation,	vert.	~5 m/s	6 s	(implicit)
	....				

--> weak stiffness in compressible models

Time integration methods

$$\frac{\partial q}{\partial t} = \mathcal{P}_s q + \mathcal{P}_f q$$

time splitting ratio:
$n_s = DT / Dt$

- Integration with small time step Δt (and e.g. additive splitting)
  inefficient, interest only in slow processes
- Semi-implicit method (<-> Semi-Lagrange-Method)
- Time-splitting method

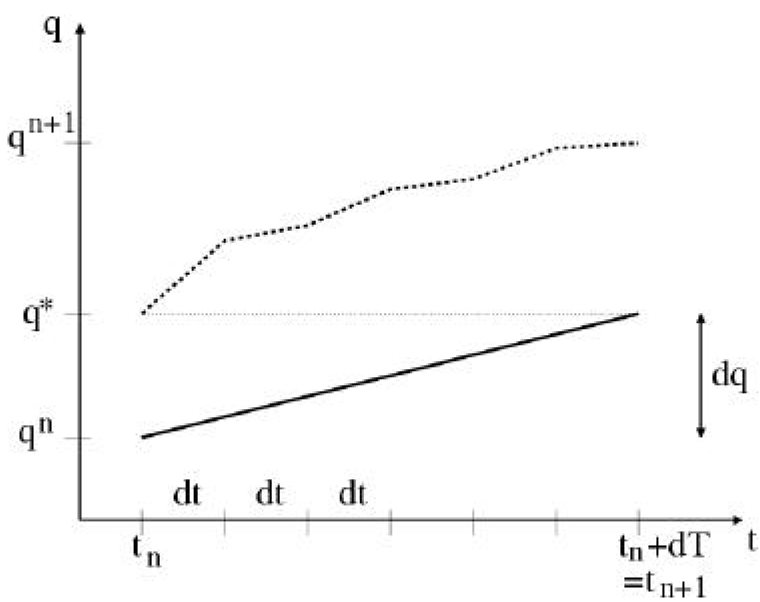
  main reason: fast processes are computationally ‚cheap'
  - Additive splitting
  - partial operator splitting
    Klemp, Wilhelmson (1978), Wicker, Skamarock, (1998, 2002), ...

Additive splitting

(complete operator splitting, method of fractional steps, multiplicative splitting)

Peaceman, Rachford (1955) (ADI-method), Marchuk (1974)

$$q^* = Q_s(q^t)$$
$$q^{t+\Delta T} = Q_f(\,Q_f(\dots Q_f(q^*)\dots))$$



- costs: 1^*Q_s, $n_s{}^*Q_f$
- stability is guarantied only if P_s and P_f both are stable *and* commutable (Leveque and Oliger, 1983)
- noisy ← only weak coupling between modes (Purser, Leslie, 1991)
- of 1. order in ΔT (for non-commutable operators)

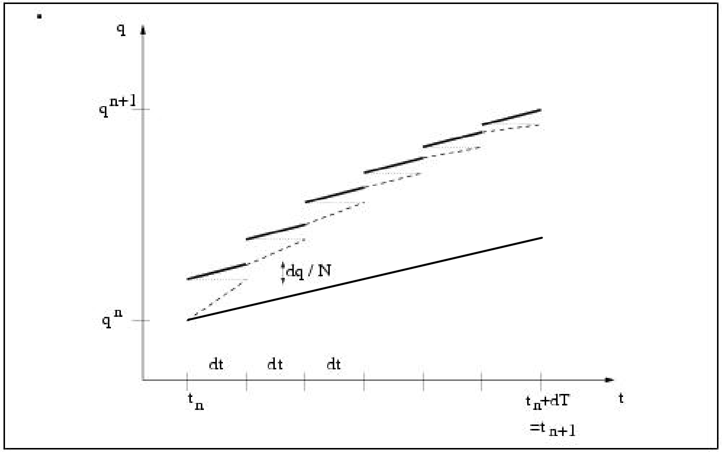
extension: Strang (1968) for non-commutable operators 2. order:

$$q^{t+\Delta T} = (Q_f)^{n/2}\, Q_s\, (Q_f)^{n/2}\, q^t$$

$\Rightarrow$ 2. order in ΔT

Time-splitted Euler-forward step

(partial operator splitting)



$$q^* = Q_s(q^t), \quad dq_s = \frac{q^* - q^t}{\Delta T}$$

and

$$\begin{aligned} q^{t+\Delta t} &= Q_f(q^t) + \Delta t \; dq_s \\ q^{t+2\Delta t} &= Q_f(q^{t+\Delta t}) + \Delta t \; dq_s \\ &\dots \\ q^{t+\Delta T} &= Q_f(q^{t+(n_s-1)\Delta t}) + \Delta t \; dq_s \end{aligned}$$

- 2-timelevel scheme
- costs: $1 \times P_s$, $n_s \times P_f$
- Skamarock, Klemp (1992), MWR: scheme is not stable (but analysis only in the time domain)
- scheme can be stabilized by divergence filtering (Baldauf (2002), COSMO-Newsletter)

Time-integration scheme

- method of lines (semi-discretization)
- time-splitted Euler-forward step can be put into arbitrary ODE-solver

ODE-solvers:

explicit 2-timelevel schemes

single stage: Euler method = forward differencing = Runge-Kutta 1. order

multistage: Runge-Kutta 2. order (e.g. Heun method, midpoint method)
Runge-Kutta 3. order
...

implicit 2-timelevel schemes

single stage: backward differencing
trapezoidal method = Crank-Nicholson

explicit 3-timelevel schemes *(occurence of computational modes)*

Leapfrog scheme
Adams-Bashforth 2. order
Magazenkov (1980)

...

but in practice: stability constraints

Klemp-Wilhelmson-time-splitting, Leapfrog-method

Klemp, Wilhelmson (1978), JAS

Leapfrog-scheme for ODE $dq/dt = f(q)$:

$$q^{n+1} = q^{n-1} + 2\Delta T \, f(q^n)$$

Leapfrog-scheme combined with splitting idea:

1. calculate the tendency of slow processes with q^t
2. perform a KW-Euler-forward-splitting from $t - \Delta T$ to $t + \Delta T$ with this slow tendency

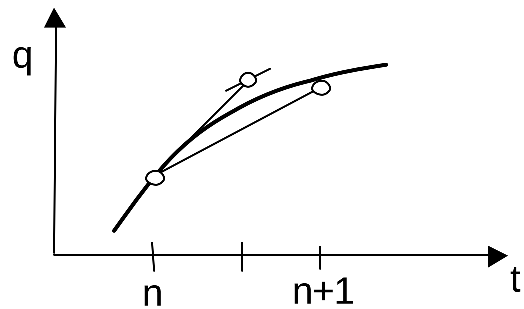
- 3-timelevel scheme
- costs: $1 \times P_s$, $2n_s \times P_f$
- disadv.: only special advection schemes are applicable
- scheme is stable with (weak) Asselin-filter and divergence damping (Skamarock and Klemp, 1992)

Time-splitting with Runge-Kutta 2. order

Wicker, Skamarock (1998), MWR

RK2-scheme (midpoint-method) for an ODE: dq/dt=f(q)

$$q^* = q^n + \frac{\Delta T}{2} f(q^n)$$
$$q^{n+1} = q^n + \Delta T \, f(q^*)$$



- 2-timelevel scheme
- Wicker, Skamarock (2002): upwind-advection stable: 3. order (C<0.88)
- combined with time-splitting-idea:
  'costs': 2* slow process, 1.5 n_s * fast process

'shortened RK2 version': first RK-step only with fast processes (Gassmann, 2004)

Runge-Kutta-Methods, order

order	number of free parameters	
1	0	Euler-forward
2	1	e.g. Heun, midpoint method
3	2	
4	2	
...	...	

general theory: Butcher (1964, ...), Butcher (1987) $\Rightarrow$

Runge-Kutta methods of 1. to 3. order: have the same order
for a single scalar equation and for a system of equations
Runge Kutta methods of order >4: can have lower order for
systems of equations than for a single equation

3. order Runge-Kutta-schemes

$$
\begin{aligned}
k_1 &= f(t^n,\ y^n), \\
k_2 &= f(t^n + \Delta t \cdot \alpha_2,\ y^n + \Delta t \cdot \beta_{21} k_1), \\
k_3 &= f(t^n + \Delta t \cdot \alpha_3,\ y^n + \Delta t \cdot (\beta_{31} k_1 + \beta_{32} k_2)), \\
y^{n+1} &= y^n + \Delta t \cdot (b_1 k_1 + b_2 k_2 + b_3 k_3)
\end{aligned}
$$

Butcher-tableau:

$$
\begin{array}{c|ccc}
0 & & & \\
\alpha_2 & \beta_{21} & & \\
\alpha_3 & \beta_{31} & \beta_{32} & \\
\hline
 & b_1 & b_2 & b_3
\end{array}
$$

6 conditions to generate a 3. order scheme:

$$
\begin{aligned}
b_1 + b_2 + b_3 &= 1 \\
\beta_{21} &= \alpha_2 \\
\beta_{31} + \beta_{32} &= \alpha_3 \\
b_2 \alpha_2 + b_3 \alpha_3 &= 1/2 \\
b_2 \alpha_2^2 + b_3 \alpha_3^2 &= 1/3 \\
b_3 \alpha_2 \beta_{32} &= 1/6
\end{aligned}
$$

RK3-Variants:

- TVD-RK3 (Shu, Osher, 1988), RK3b (Hundsdorfer et al., 1995)

$$
\begin{aligned}
y^* &= y^n + \Delta t\, f(t^n,\ y^n), \\
y^{**} &= \frac{3}{4} y^n + \frac{1}{4} y^* + \frac{1}{4} \Delta t\, f(t^n + \Delta t,\ y^*) \\
y^{n+1} &= \frac{1}{3} y^n + \frac{2}{3} y^{**} + \frac{2}{3}\, \Delta t\, f(t^n + \frac{1}{2}\, \Delta t,\ y^{**})
\end{aligned}
$$

is used in LMK
(Förstner, Doms, 2004)

- RK3a (Hundsdorfer et al., 1995)
- minimal storage scheme (Williamson, 1980)

- Wicker, Skamarock (2002) is *not* a 3. order RK-scheme

$$
\begin{aligned}
y^* &= y^n + \frac{1}{3} \Delta t\, f(t^n,\ y^n), \\
y^{**} &= y^n + \frac{1}{2} \Delta t\, f(t^n + \frac{1}{3} \Delta t,\ y^*) \\
y^{n+1} &= y^n + \Delta t\, f(t^n + \frac{1}{2}\, \Delta t,\ y^{**})
\end{aligned}
$$

Advection-schemes of higher order

$$\frac{q_i^{n+1} - q_i^n}{\Delta t} = -\frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x}$$

Fluxes of 3rd to 6th order (from spatial discretization of advection operator)

$$\begin{aligned}
F_{i-1/2}^{(3)} &= F_{i-1/2}^{(4)} - \frac{|u_{i-1/2}|}{12}\left(3(q_i - q_{i-1}) - (q_{i+1} - q_{i-2})\right) \\
F_{i-1/2}^{(4)} &= \frac{u_{i-1/2}}{12}\left(7(q_i + q_{i-1}) - (q_{i+1} + q_{i-2})\right) \\
F_{i-1/2}^{(5)} &= F_{i-1/2}^{(6)} - \frac{|u_{i-1/2}|}{60}\left(10(q_i - q_{i-1}) - 5(q_{i+1} - q_{i-2}) + (q_{i+2} - q_{i-3})\right) \\
F_{i-1/2}^{(6)} &= \frac{u_{i-1/2}}{60}\left(37(q_i + q_{i-1}) - 8(q_{i+1} + q_{i-2}) + (q_{i+2} + q_{i-3})\right)
\end{aligned}$$

- Hundsdorfer et al. (1995), JCP
- Wicker, Skamarock (2002), MWR

stable Courant-numbers for advection schemes (Wicker, Skamarock, 2002)

	1.	2.	3.	4.	5.	6.
Leapfrog	0	1	0	0.72	0	0.62
Euler	1	0	0	0	0	0
RK2	1	0	0.87	0	0 !	0
RK3	1.25	1.73	1.62	1.26	1.43	1.09
RK4	1.39	2.82	1.74	2.06	1.73	1.78

All 2. order Runge-Kutta-schemes have the same linear stability properties for c=const. !

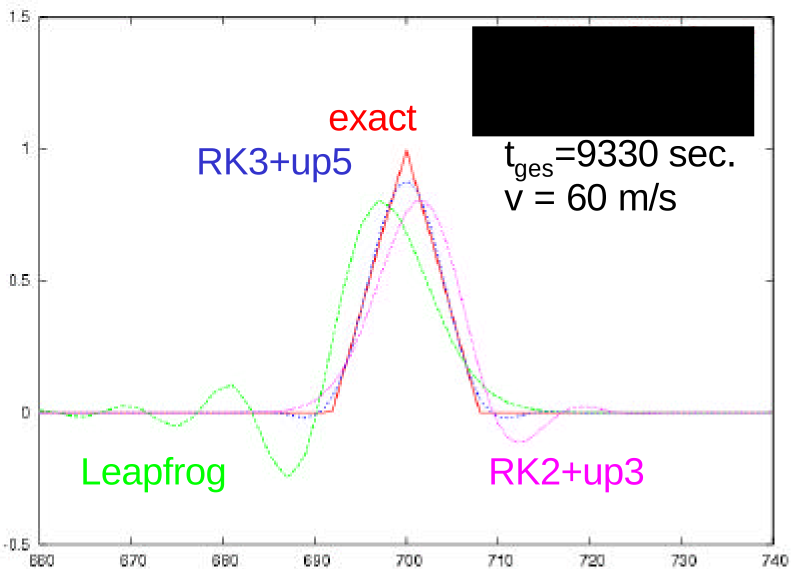
All 3. order Runge-Kutta-schemes have the same linear stability properties for c=const. !
(holds also for Wicker, Skamarock (2002) ‚RK3‘-variant !)

Horizontal advection in time splitting schemes

- Leapfrog + centered diff. 2. order (currently used LM/LME) (C < 1)
- Runge-Kutta 2. order $O(\Delta t^2)$ + upwind 3. order $O(\Delta x^3)$ (C < 0.87)
- Runge-Kutta 3. order $O(\Delta t^3)$ + upwind 5. order $O(\Delta x^5)$ (C < 1.43)

(Wicker, Skamarock, 2002)

[redacted]



advection equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

Courant number $C = v * \Delta t / \Delta x$

Linearised, 2-dim. Sound-Buoyancy-Advection-System

$$
\begin{array}{rcllll}
\dfrac{\partial u}{\partial t} + U_0 \dfrac{\partial u}{\partial x} &=& -\dfrac{1}{\rho_0}\dfrac{\partial p'}{\partial x} & & & + Q_x \\
\dfrac{\partial w}{\partial t} + U_0 \dfrac{\partial w}{\partial x} &=& -\dfrac{1}{\rho_0}\dfrac{\partial p'}{\partial z} & & -\left(\dfrac{p'}{p_0} - \dfrac{T'}{T_0}\right) g & + Q_z \\
\dfrac{\partial p'}{\partial t} + U_0 \dfrac{\partial p'}{\partial x} &=& -\dfrac{c_p}{c_v} p_0 \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z}\right) & & + \rho_0 g w & \\
\dfrac{\partial T'}{\partial t} + U_0 \dfrac{\partial T'}{\partial x} &=& -\dfrac{R}{c_v} T_0 \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z}\right) & & -\dfrac{\partial T_0}{\partial z} w & \\
\text{Tend.} \quad \text{Adv.} & & \text{Sound} & & \text{Buoyancy} &
\end{array}
$$

- primitive variables

$$
u, \quad w, \quad p = p_0 + p', \quad T = T_0 + T'
$$

- Q_x, Q_z e.g. a divergence damping

$$
Q_x = \alpha_{div} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right), \quad Q_z = \alpha_{div} \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)
$$

Von-Neumann stability analysis

Linearized PDE-system for u(x,z,t), w(x,z,t), ... with constant coefficients
Discretization u^n_{jl}, w^n_{jl}, ... (grid sizes Δx, Δz)
single Fourier-Mode:

$$u^n_{jl} = u^n \exp(i\, k_x\, j\, \Delta x + i\, k_z\, l\, \Delta z)$$

2-timelevel schemes:

$$\begin{pmatrix} u^{n+1} \\ w^{n+1} \\ p'^{n+1} \\ T'^{n+1} \end{pmatrix} = Q \begin{pmatrix} u^{n} \\ u^{n} \\ p'^{n} \\ T'^{n} \end{pmatrix}$$

Determine eigenvalues λ_i of Q
scheme is stable, if $\max_i |\lambda_i| \leq 1$

find λ_i analytically or numerically by scanning $k_x \Delta x = -\pi..+\pi$, $k_z \Delta z = -\pi..+\pi$

Sound

- temporal discret.: 'generalized' Crank-Nicholson
  β=1: implicit, β=0: explicit
- spatial discret.: centered diff.

$$
\begin{aligned}
\frac{u^{n+1}-u^n}{\Delta t} &= -\frac{1}{\rho_0}\left(\beta_1^s\,\delta_x p'^{n+1} + (1-\beta_1^s)\,\delta_x p'^{n}\right)\\
\frac{w^{n+1}-w^n}{\Delta t} &= -\frac{1}{\rho_0}\left(\beta_2^s\,\delta_z p'^{n+1} + (1-\beta_2^s)\,\delta_z p'^{n}\right)\\
\frac{p^{n+1}-p^n}{\Delta t} &= -\frac{c_p}{c_v}p_0\left(\beta_3^s\,\delta_x u^{n+1} + (1-\beta_3^s)\,\delta_x u^{n} + \beta_4^s\,\delta_z w^{n+1} + (1-\beta_4^s)\,\delta_z w^{n}\right)\\
\frac{T^{n+1}-T^n}{\Delta t} &= -\frac{R}{c_v}T_0\left(\beta_5^s\,\delta_x u^{n+1} + (1-\beta_5^s)\,\delta_x u^{n} + \beta_6^s\,\delta_z w^{n+1} + (1-\beta_4^s)\,\delta_z w^{n}\right)
\end{aligned}
$$

Courant-numbers: $C_{snd,x} = c_s\frac{\Delta t}{\Delta x}$, $C_{snd,z} = c_s\frac{\Delta t}{\Delta z}$ $c_s^2 = \frac{c_p}{c_v}\frac{p_0}{\rho_0}$

fully explicit	uncond. unstable	-	
forward-backward (Mesinger, 1977), unstaggered grid	stable for $C_x^2+C_z^2<2$	neutral	4 dx, 4dz
forward-backward, staggered grid	stable for $C_x^2+C_z^2<1$	neutral	2 dx, 2dz
forward-backw.+vertically Crank-Nic. ($\beta_{2,4,6}=1/2$)	stable for $C_x<1$	neutral	2 dx
forward-backw.+vertically Crank-Nic. ($\beta_{2,4,6}>1/2$)	stable for $C_x<1$	damping	2 dx

Divergence damping

$$\frac{\partial \mathbf{v}}{\partial t} + \ldots = \ldots + \alpha_{div} \nabla(\operatorname{div} \mathbf{v})$$

$\Rightarrow$ Diffusion of Divergence

Courant-numbers: $C_{div,x} := \alpha_{div} \frac{\Delta t}{\Delta x^2}, \quad C_{div,z} := \alpha_{div} \frac{\Delta t}{\Delta z^2}$

1D, explicit	stable for $C_{div,x} < \frac{1}{2}$
2D, explicit (staggered grid)	stable for $C_{div,x} + C_{div,z} < \frac{1}{2}$
2D, implicit	unconditionally stable
2D, vertically implicit	stable for $C_{div,x} < \frac{1}{2}$, $C_{div,z}$ arbitr.

Wicker, Skamarock (2002): $\alpha_{div} \approx 0.1\, c_s^2\, \Delta t$ $\Rightarrow$ for LMK: $\alpha_{div} \approx 50000 m^2/s$

What is the influence of

- **different time-splitting schemes**
    - Euler-forward
    - Runge-Kutta 2. order
    - Runge-Kutta 3. order (WS2002)
- **and smoothing** (4. order horizontal diffusion) **?**
    - $K_{smooth}\ \Delta t\ /\ \Delta x^4 = 0\ \ /\ \ 0.05$

- fast processes (with operatorsplitting)
    - sound (Crank-Nic., β=0.6),
    - divergence-damping (vertical implicit, C_{div}=0.1)
    - no buoyancy
- slow process: upwind 5. order
- aspect ratio: $\Delta x\ /\ \Delta z$=10
- $\Delta T\ /\ \Delta t$=12

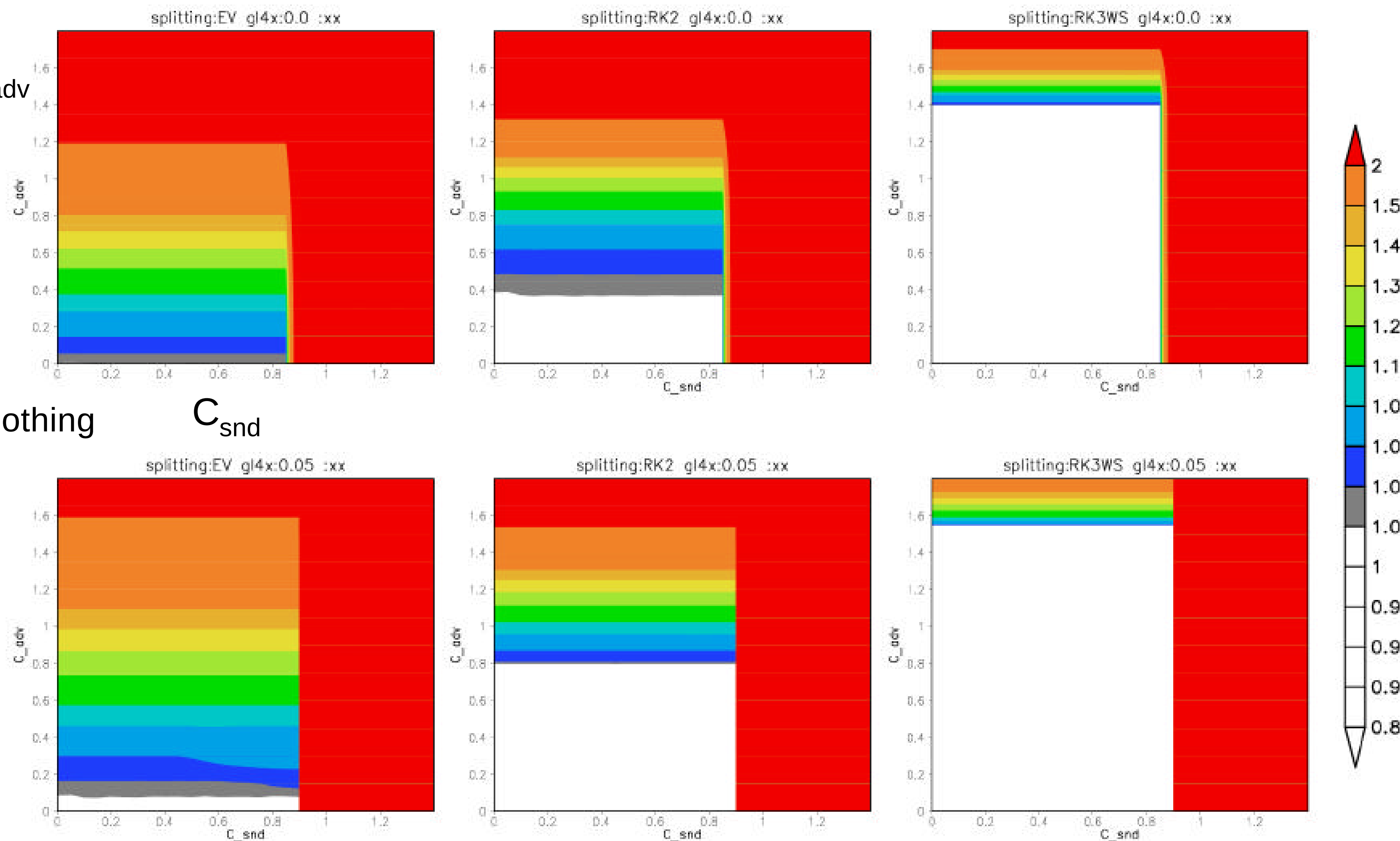
C_{adv}

no

smoothing

C_{snd}

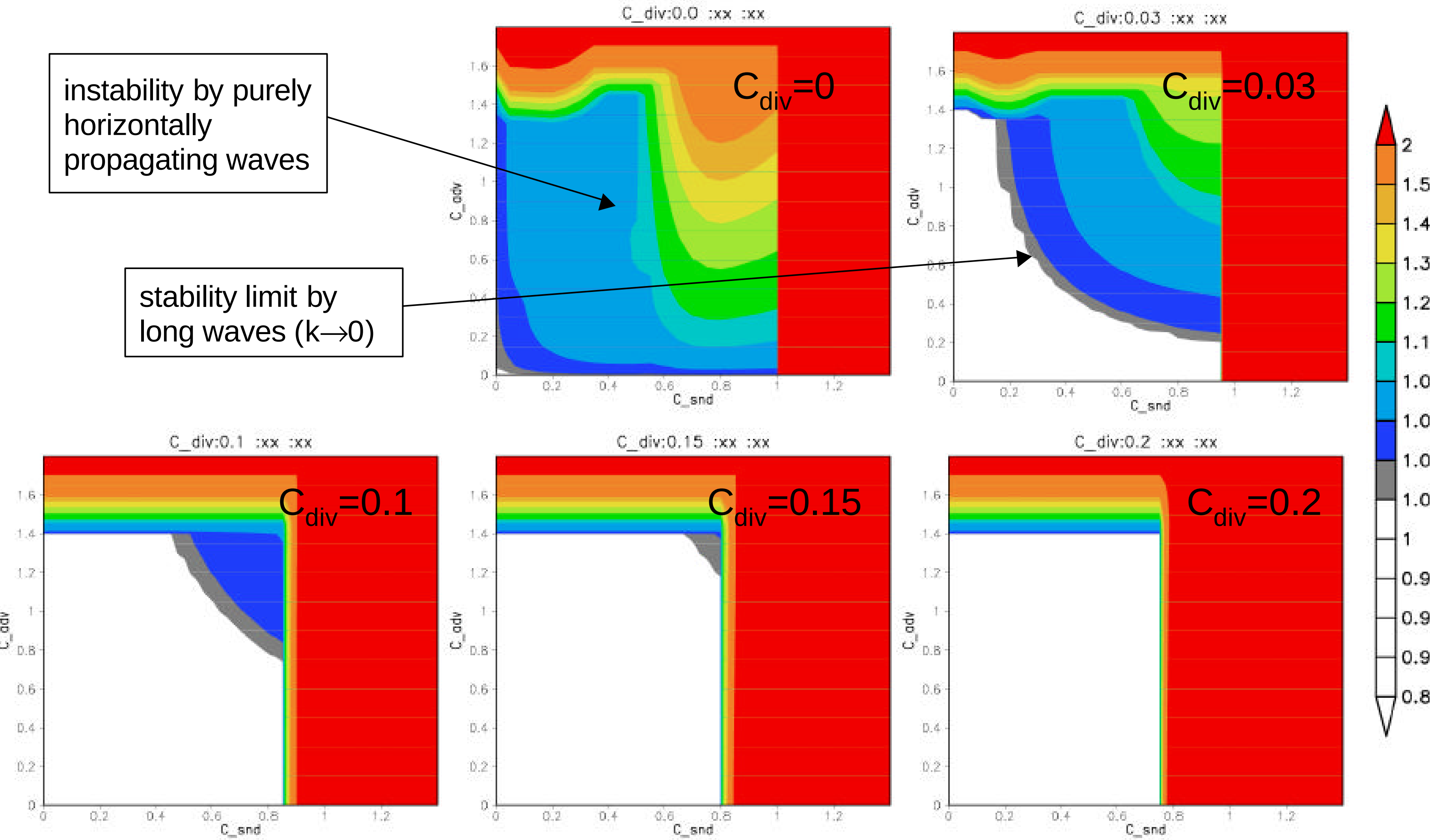
ves



Euler-forward | Runge-Kutta 2. order | Runge-Kutta 3. order

What is the influence of divergence filtering ?

- fast processes (operatorsplitting):
    - sound (Crank-Nic., β=0.6),
    - divergence damping (vertical implicit)
    - no buoyancy
- slow process: upwind 5. order
- time splitting RK 3. order (WS2002-Version)
- aspect ratio: Δx / Δz=10
- ΔT / Δt=6


instability by purely horizontally propagating waves
stability limit by long waves (k→0)
C_div:0.0 :xx :xx
C_{div}=0
C_div:0.03 :xx :xx
C_{div}=0.03
C_div:0.1 :xx :xx
C_{div}=0.1
C_div:0.15 :xx :xx
C_{div}=0.15
C_div:0.2 :xx :xx
C_{div}=0.2
C_adv
C_snd

Buoyancy terms:

$$
\begin{aligned}
\frac{w^{n+1} - w^n}{\Delta t} &= g \left(\beta_T^b \frac{T'^{n+1}}{T_0} + (1-\beta_T^b) \frac{T'^n}{T_0} - \beta_p^b \frac{p'^{n+1}}{p_0} - (1-\beta_p^b) \frac{p'^n}{p_0} \right) \\
\frac{p'^{n+1} - p'^n}{\Delta t} &= \rho_0 g (\beta_3^b \, w^{n+1} + (1-\beta_3^b) \, w^n) \\
\frac{T'^{n+1} - T'^n}{\Delta t} &= -\frac{\partial T_0}{\partial z} (\beta_4^b \, w^{n+1} + (1-\beta_4^b) \, w^n)
\end{aligned}
$$

acoustic cut-off frequency $\omega_a := \sqrt{N^2 + \frac{g^2}{c_S^2}}$; $C_{buoy} := \omega_a \, \Delta t$

$C_\beta = \frac{1}{T_0} \frac{\partial T_0}{\partial z} \frac{c_S^2}{g} \approx -0.24$ (standard atmosphere)

fully explicit	unstable	-
forward-backward	stable for $C_{buoy} < 2$	neutral
Crank-Nicholson $\beta = 1/2$	uncond. stable	neutral
Crank-Nicholson $\beta > 1/2$	uncond. stable	damping
implicit	uncond. stable	damping

Commutation with other numerical operators:

$Q_{buoy} \cdot Q_{sound} \neq Q_{sound} \cdot Q_{buoy}$, $\qquad Q_{buoy} \cdot Q_{div} \neq Q_{div} \cdot Q_{buoy}$

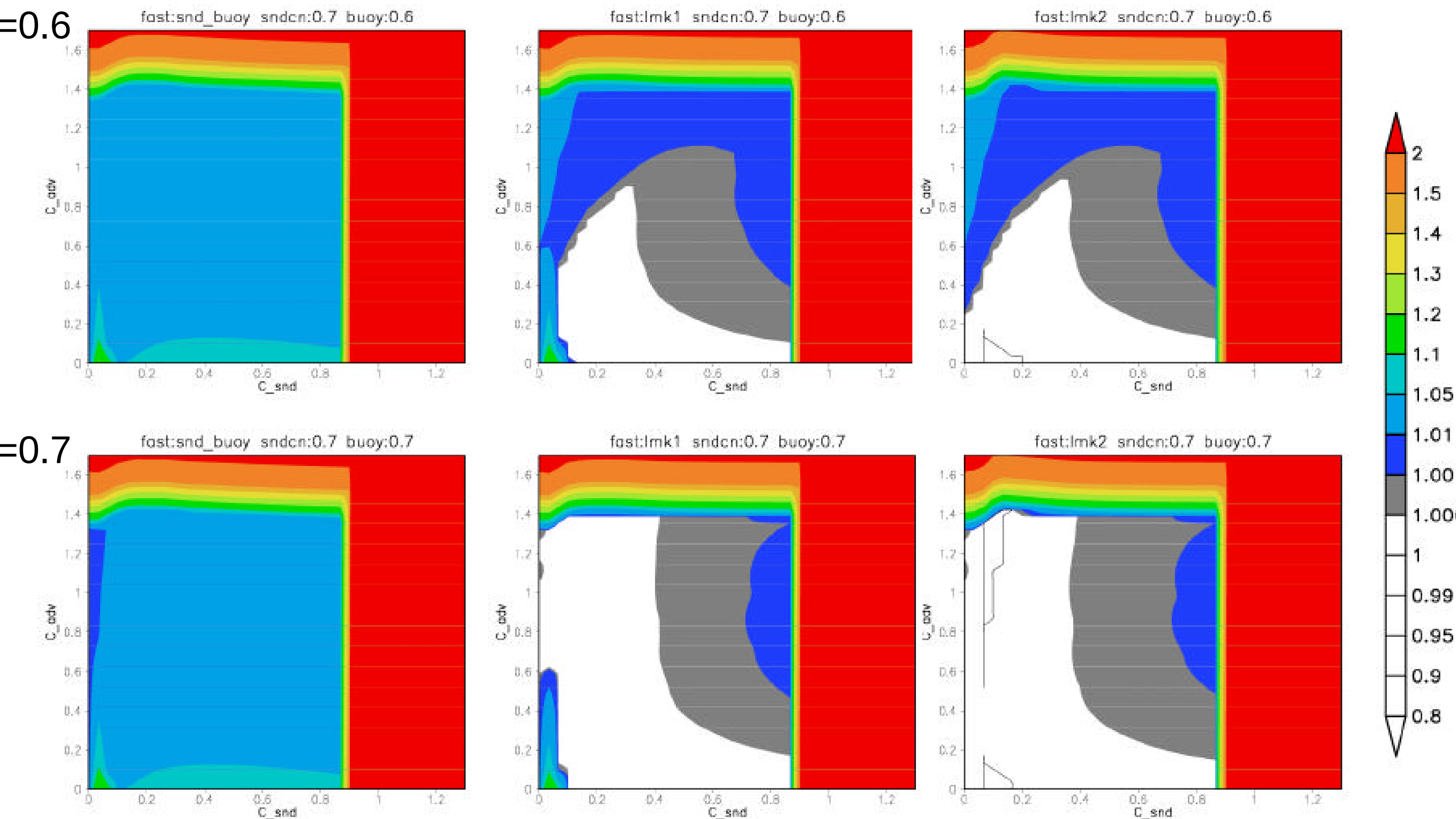
How to handle the fast processes with buoyancy?

with buoyancy ($C_{buoy} = \omega_a dt = 0.15$, standard atmosphere)

- different fast processes:
  - operatorsplitting (Marchuk-Splitting): ‘Sound -> Div. -> Buoyancy‘
  - partial adding of tendencies: ‘(Sound+Buoyancy) -> Div.')
  - adding of all fast tendencies: ‘Sound+Div.+Buoyancy‘
- different Crank-Nicholson-weights for buoyancy:

  β=0.6 / 0.7

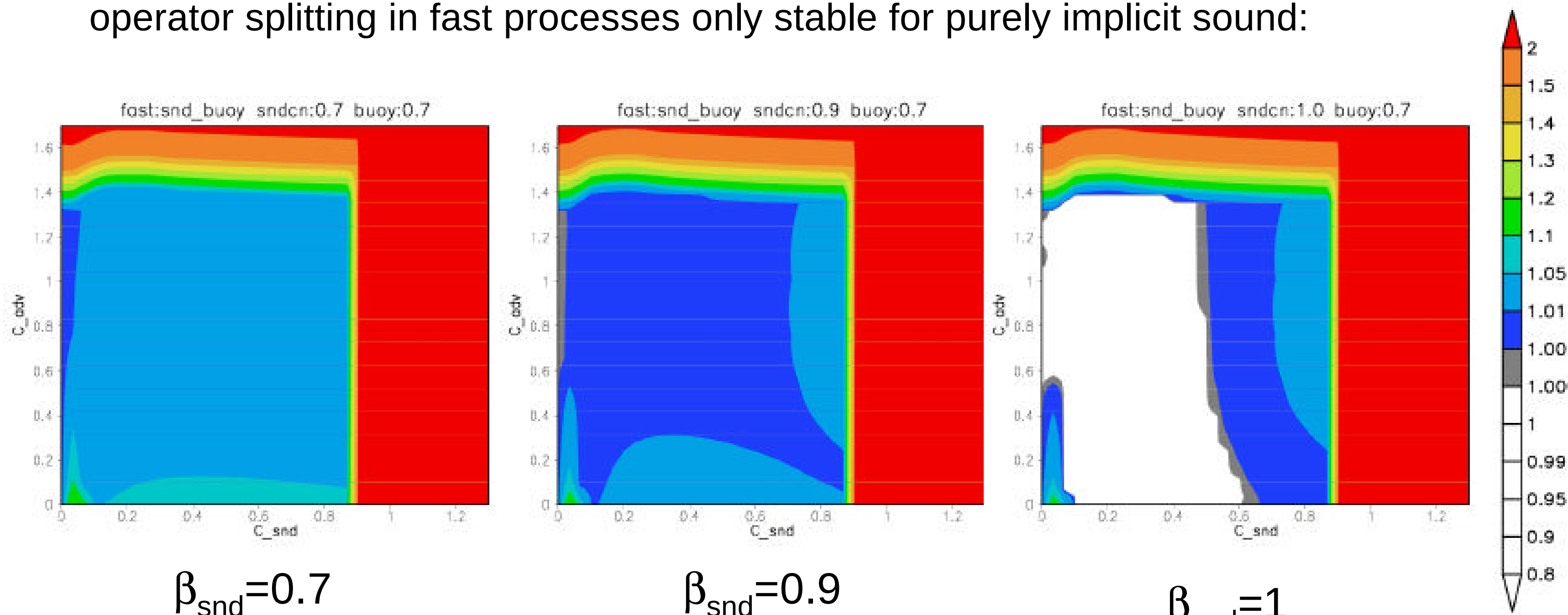
- RK3-scheme
- slow process: upwind 5. order
- aspect ratio: dx/dz=10
- dT/dt=6



'Sound -> Div. -> Buoyancy' '(Sound+Buoyancy) -> Div.') 'Sound+Div.+Buoyancy'

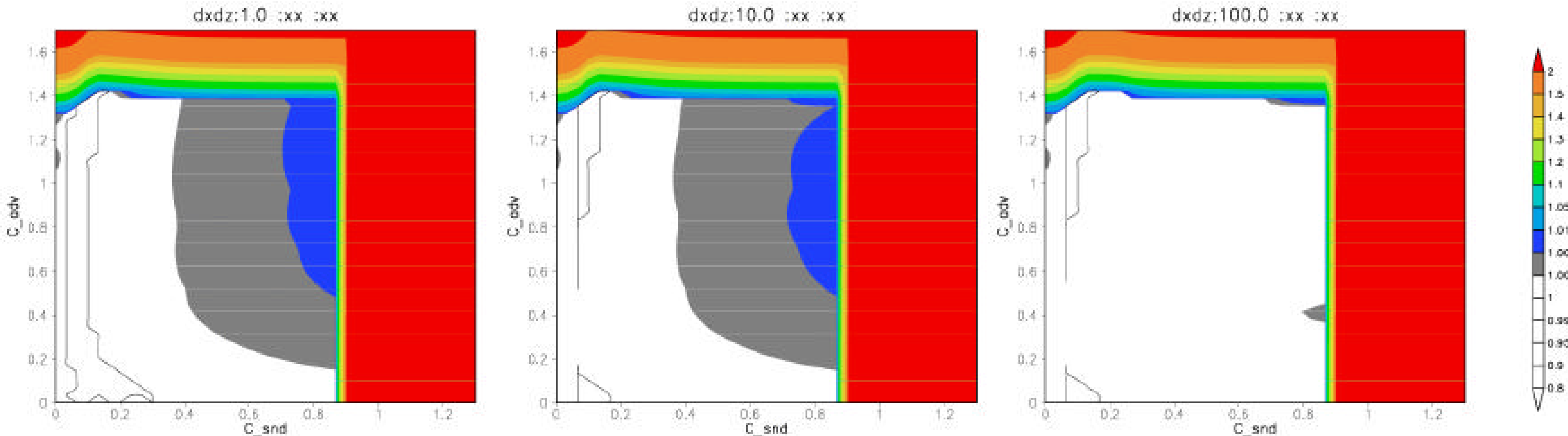
curious result:
operator splitting of all the fast processes is <u>not</u> the best choice,
better: simple addition of tendencies.

operator splitting in fast processes only stable for purely implicit sound:



β_{snd}=0.7 β_{snd}=0.9 β_{snd}=1 implicit

What is the influence of the grid anisotropy?



Δx:Δz=1 Δx:Δz=10 Δx:Δz=100

Conclusions from stability analysis of the 2-timelevel splitting schemes

- KW-RK2 allows only smaller time steps with upwind 5. order than RK3 → RK3 is more accurate and more efficient (!) than RK2
- Divergence filtering is needed ($C_{div,x}$ = 0.1: good choice) to stabilize purely horizontal waves.
  Even KW-Euler-Forward-scheme can be stabilized by a (strong) divergence damping (stability analysis by Skamarock, Klemp (1992) too carefully)
- bigger $\Delta x/\Delta z$ seems not to be problematic for stability
- increasing $\Delta T/\Delta t$ does not reduce stability
  prove for no stability constraint for n_s in the 1D sound-advection-system
- buoyancy in fast processes: better addition of tendencies than operator splitting (operator splitting needs purely implicit scheme for the sound)
  in case of stability problems: reduction of small time step recommended

Test of the dynamical core: linear, hydrostatic mountain wave

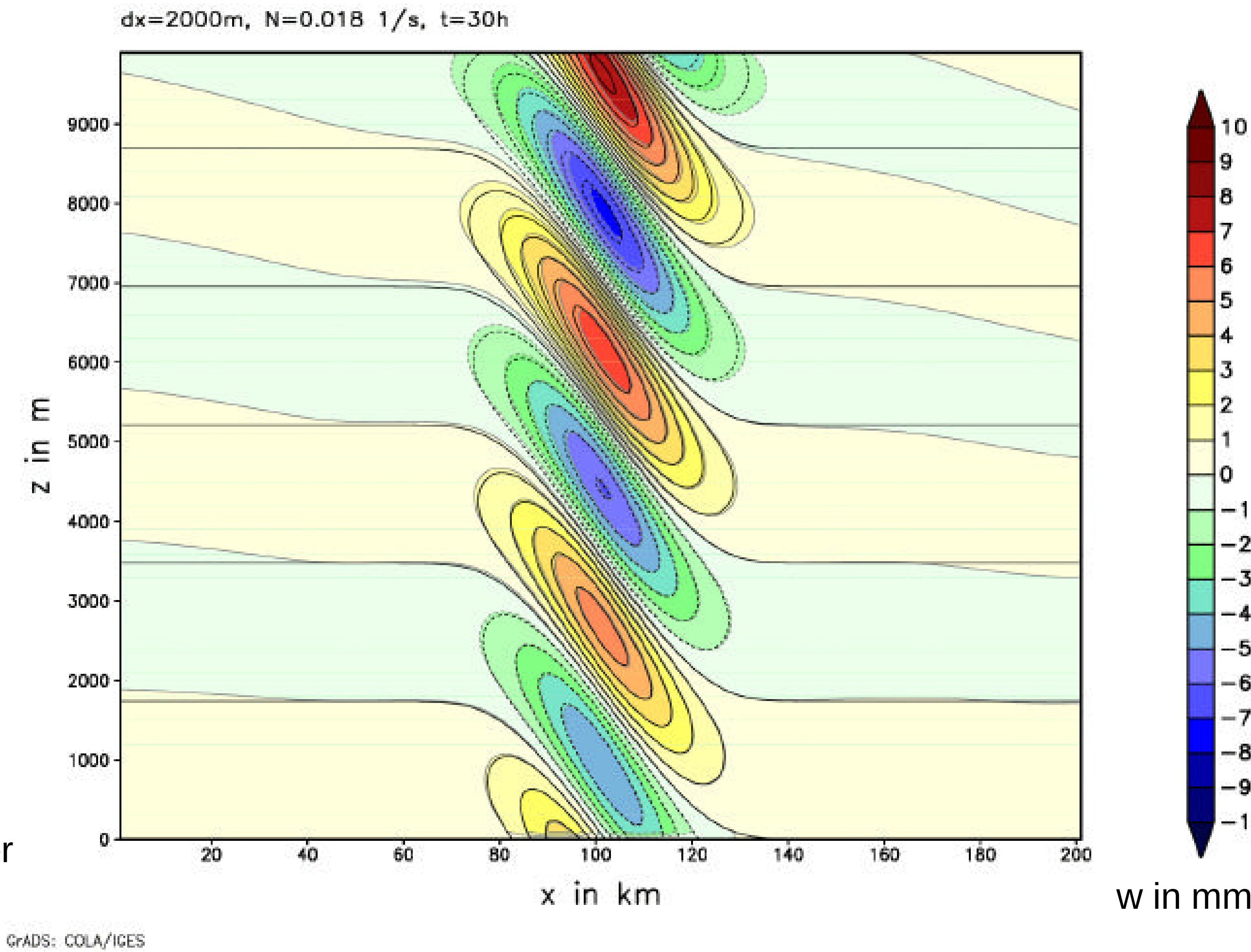
Gaussian hill
Half width = 40 km
Height = 10 m
U0 = 10 m/s
isothermal stratification

dx=2 km
dz=100 m
T=30 h

analytic solution:
black lines

simulation:
colours + grey lines

RK 3. order + upwind 5. order



Question: can time splitting schemes deliver a correct stationary solution?

Example: **scalar relaxation equation with external force** (Murthy, Nanjundiah, 2000)

$$\frac{d\phi}{dt} = \underbrace{-\beta\phi}_{Relax.} + \underbrace{g}_{Force}, \quad \beta > 0,\ \beta, g \text{ constant}$$

stationary solution:

$$\phi_s = \frac{g}{\beta}$$

Discretisation of single processes with simple Euler-forward-schemes; resulting stationary solution of the numerical time splitting system:

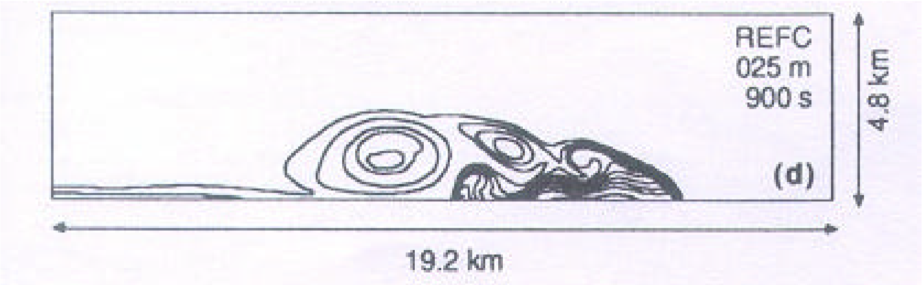
Splitting	Relax. fast, Forcing slow	Forcing fast, Relax. slow
Additive	$g/\beta \cdot \left(1 + \frac{1}{2}\frac{n_s-1}{n_s}\beta\,\Delta t + \ldots\right)$	$g/\beta \cdot (1 - \beta\,\Delta T)$
KW-EV	g/β	g/β
KW-Leapfrog	g/β	g/β
KW-RK2	g/β	g/β
KW-RK2-short	g/β	$g/\beta \cdot (1 - \beta\frac{\Delta T}{2})$
KW-'RK3'	g/β	g/β
KW-RK3b	g/β	g/β

consistent solutions for $\Delta T, \Delta t \rightarrow 0$, but in practice: $\beta\,\Delta T \approx 1$ or $\beta\,\Delta t \approx 1$

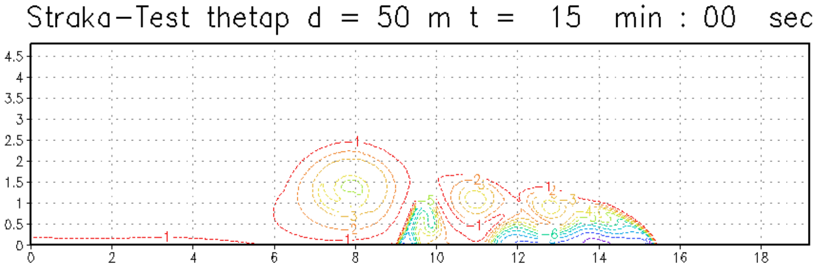
(Baldauf, 2004)

Test of the dynamical core: density current (Straka et al., 1993)

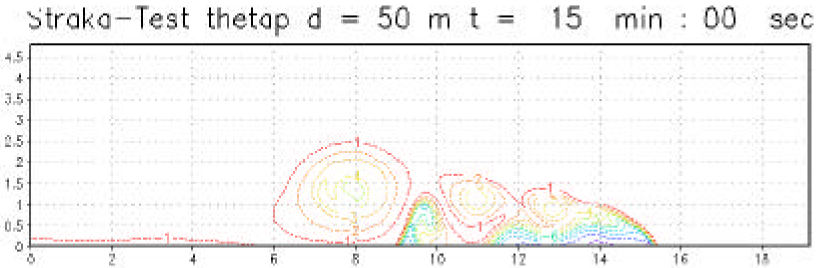
θ' after 900 s. (Reference)
by Straka et al. (1993)



RK3 + upwind 5. order



RK2 + upwind 3. order



Implicit Vertical Advection for dynamic variables (u, v, w, T, p‘)

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial z} = M$$

Generalized Crank-Nicholson-advection

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{c_j}{\Delta z} \cdot (\underbrace{\beta\, A_j^{n+1}}_{implicit} + \underbrace{(1-\beta)\, A_j^n}_{explicit}) = M_j^n \qquad C_j = c_j \frac{\Delta t}{\Delta z}$$

(Dimensionless) Advection operator for centered differences 2. order (3-point-stencil):

$$A_j^n = \frac{1}{2} f_{j+1}^n - \frac{1}{2} f_{j-1}^n$$

® Lin. eq. system with a tridiagonal matrix, needs ~3 N operations

Motivation for a better scheme:
explicitly resolved convection, higher values of w

dim.less advection operator for upwind 3. order (4-point stencil)

$$A_j^n := \left(\frac{1}{6} + \frac{C_j^2}{12}\right) f_{j-2}^n + \left(-1 - \frac{C_j^2}{4}\right) f_{j-1}^n + \left(\frac{1}{2} + \frac{C_j^2}{4}\right) f_j^n + \left(\frac{1}{3} - \frac{C_j^2}{12}\right) f_{j+1}^n$$

case $C_j > 0$

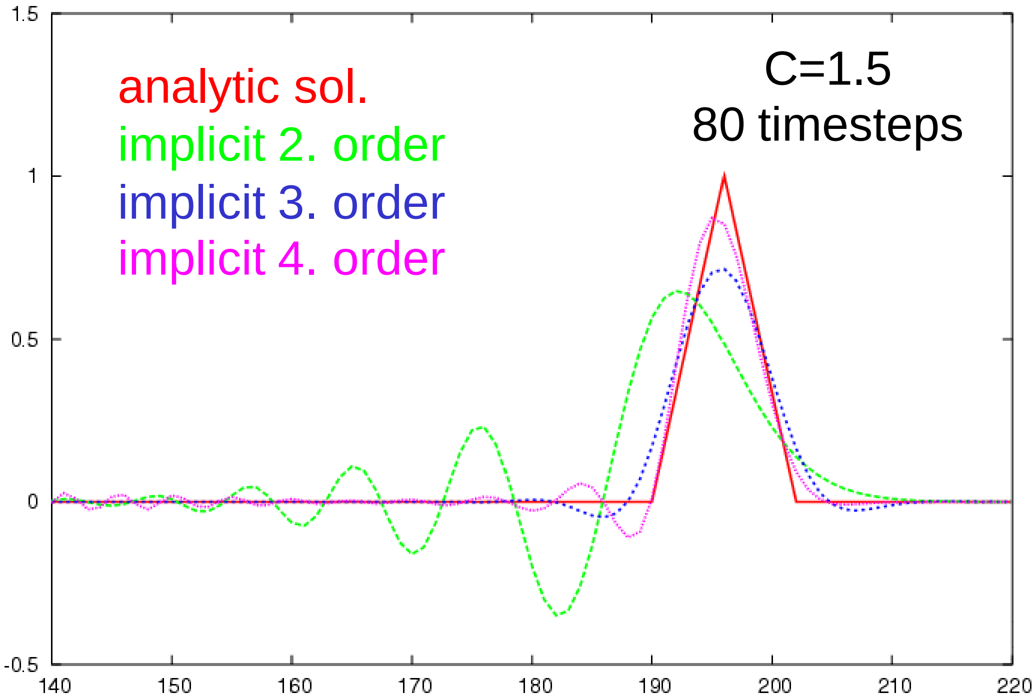
([1] Crowley 3. order, e.g. Tremback et al., 1987)

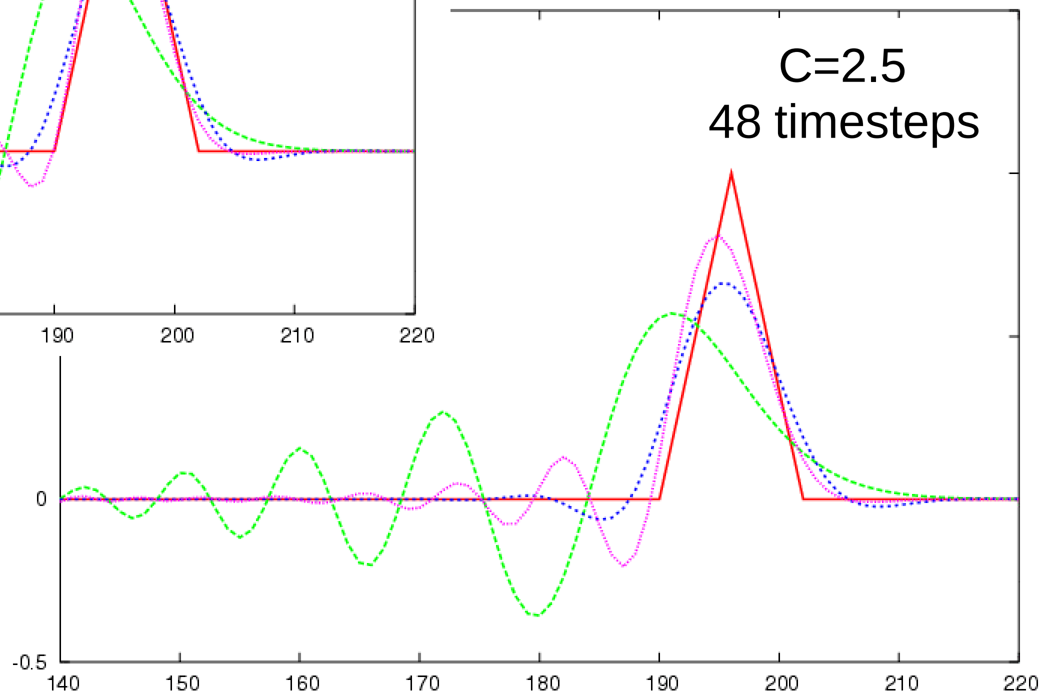
- β=1/2: unconditionally stable, damping, truncation error 3. order
- β>1/2: unconditionally stable, damping, trunc. error 1. order
- β<1/2: unstable

Lin. eq. system with a 5-band diagonal matrix
needs ~14 N operations

LMK: Subr. ‚complete_tendencies_uvwTpp_CN3‘

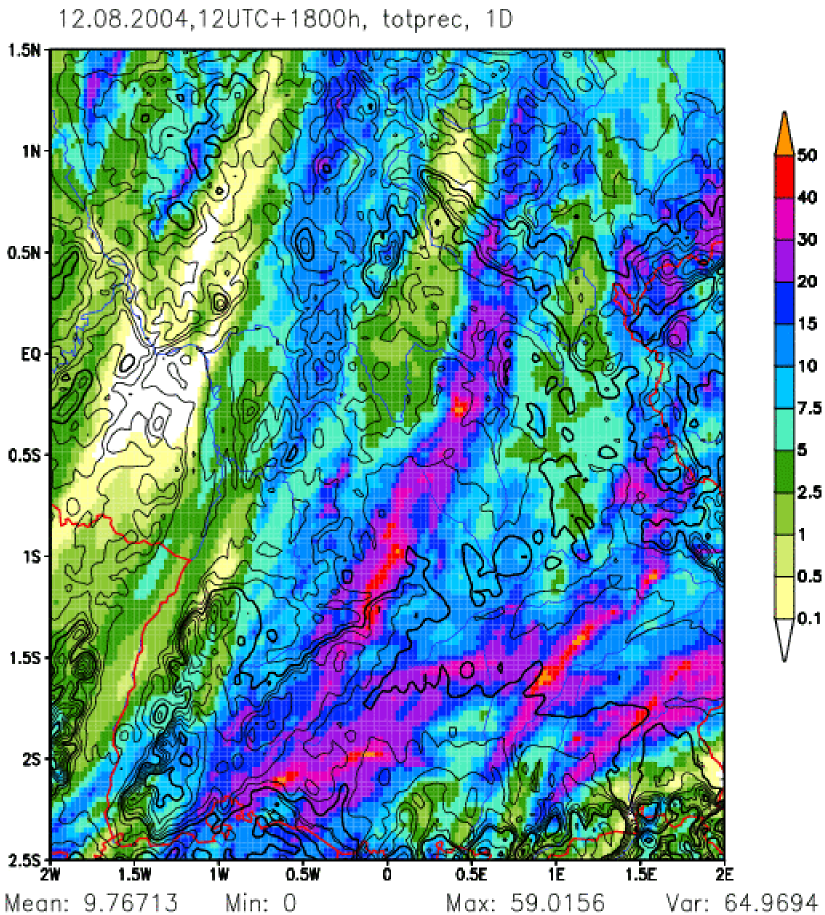
Idealized 1D advection test



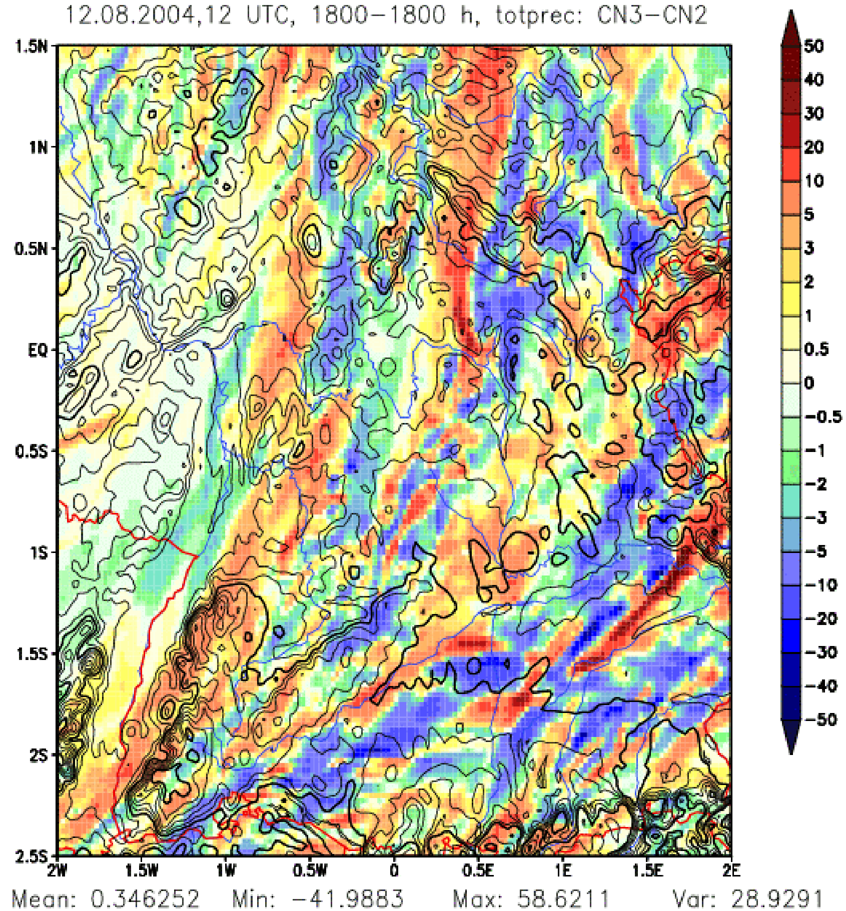


Real case study: LMK (2.8 km resolution) ‚12.08.2004, 12UTC-run'

implicit vertical adv. 2. order

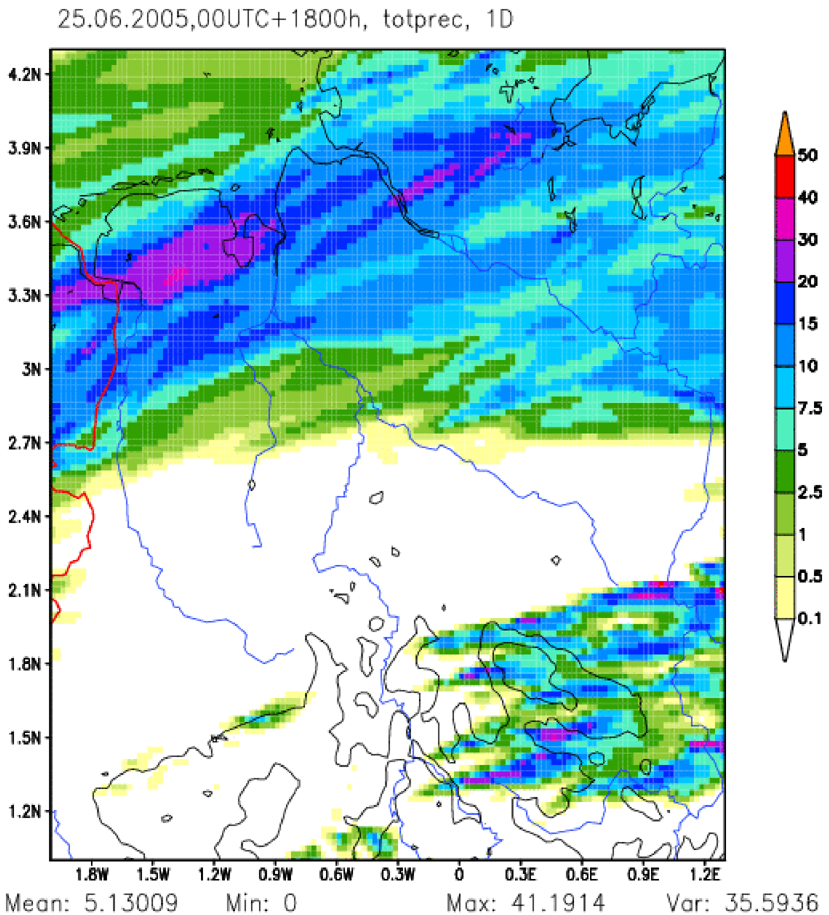


difference: 3. order - 2. order

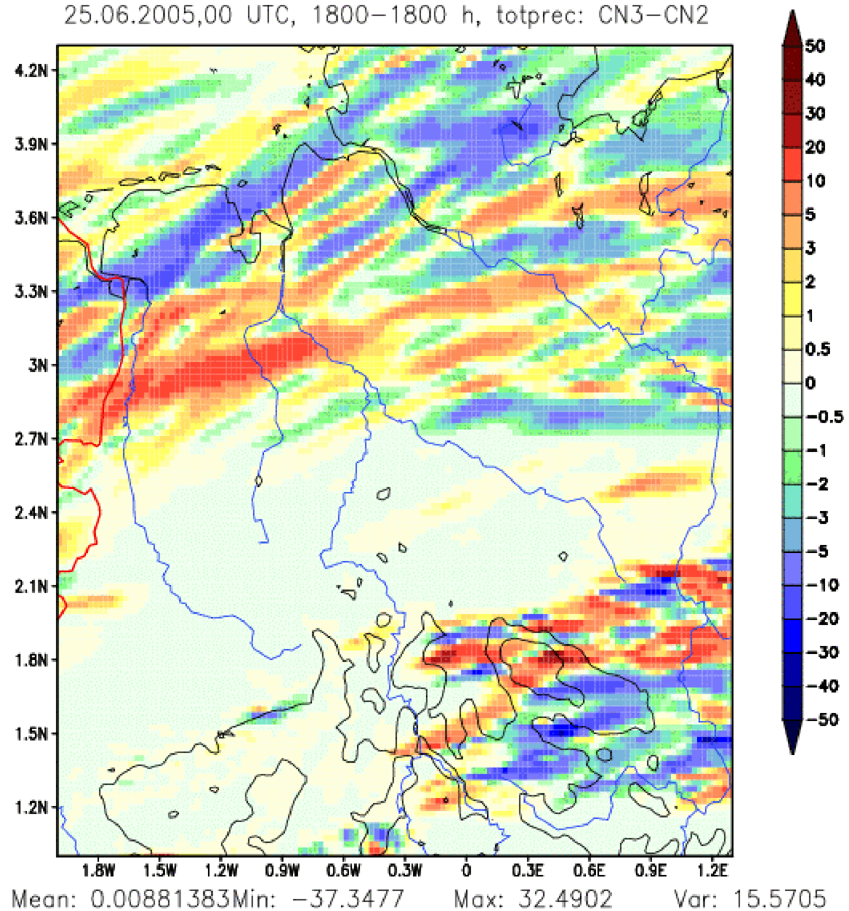


Real case study: LMK (2.8 km resolution), 25.06.2005, 00UTC-run‘

implicit vertical adv. 2. order



difference: 3. order - 2. order



2-dim. horizontal Advection

- 2D-advection in RK-schemes by a simple adding of tendencies (operator splitting (e.g. corner transport upstream (CTU) method) is not possible for upstream 3., 5., ... order)
- this is limited by

$$|C_x| + |C_y| < \text{const.}$$

can be proven for RK2 + upwind 3. order
it holds empirically also for RK3 + upwind 5. order

- compare with the usual formulated 2-dimensional stability criterion:

$$\Delta t < \frac{\Delta s}{\sqrt{2}\,|\mathbf{v}|} \quad \Rightarrow \quad |\mathbf{C}| < \frac{1}{\sqrt{2}}$$

Physics-Dynamics-coupling

Descr. of Advanced Research WRF, v. 2 (2005)

fields $\varphi^n=(u, v, w, pp, T, ...)^n$

Physics (I)
- Radiation
- Shallow Convection
- Coriolis force
- Turbulence

‚Physics (I)'-tendencies $\Delta\varphi^n_{(phys\ I)}$

$+ \Delta\varphi^{n-1}_{(phys\ II)}$ **NEW!**

Dynamics
Runge-Kutta [$\Delta\varphi_{(phys)} + \Delta\varphi_{(adv)}$ --> fast_waves]

fields $\varphi^*=(u, v, w, pp, T, ...)^*$

$- \Delta\varphi^{n-1}_{(phys\ II)}$ **NEW!**

Physics (II)
- Cloud Physics

‚Physics (II)'-tendencies $\Delta\varphi^n_{(phys\ II)}$

fields $\varphi^{n+1}=(u, v, w, pp, T, ...)^{n+1}$

Dynamic Bottom Boundary...

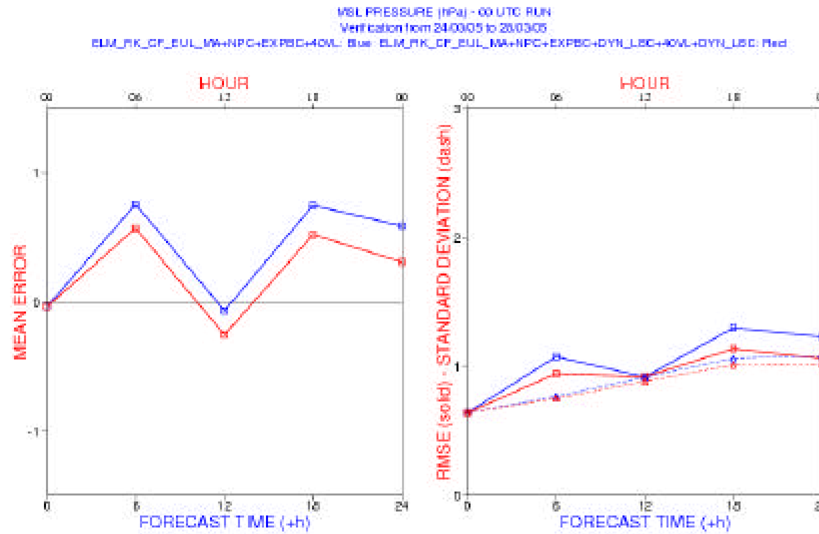
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho a \cos\varphi}\left(\frac{\partial p^*}{\partial \lambda} + \frac{J_\lambda}{\sqrt{G}}\frac{\partial p^*}{\partial \zeta}\right) + f_u,$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho a}\left(\frac{\partial p^*}{\partial \varphi} + \frac{J_\varphi}{\sqrt{G}}\frac{\partial p^*}{\partial \zeta}\right) + f_v,$$

$$\frac{\partial w}{\partial t} = \frac{1}{\rho\sqrt{G}}\frac{\partial p^*}{\partial \zeta} + g\frac{\rho_0}{\rho}\left(\frac{T-T_0}{T} - \frac{T_0 p^*}{T p_0}\right) + f_w.$$

... Condition...

$$\frac{\partial \dot\zeta}{\partial t} = \frac{1}{\sqrt{G}}\left(\frac{1}{a\cos\varphi}\frac{\partial z}{\partial \lambda}\frac{\partial u}{\partial t} + \frac{1}{a}\frac{\partial z}{\partial \varphi}\frac{\partial v}{\partial t} - \frac{\partial w}{\partial t}\right) = 0$$

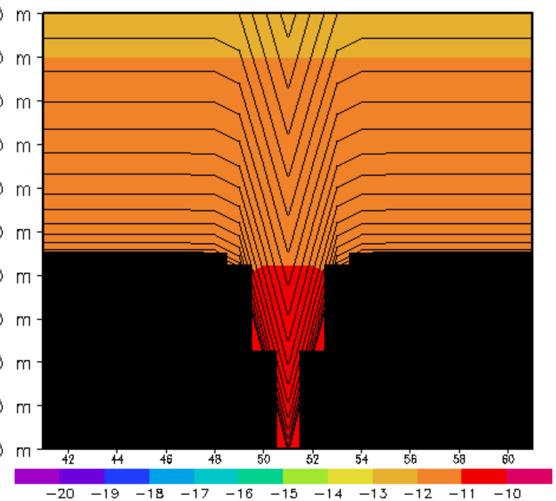


"(Positive) Pressure Bias Problem"
blue: Old Bottom Boundary Cond.
red: Dynamic Bottom Boundary Cond.
(Figures by Torrisi, CNMCA Rom)

... for metric pressure gradient term in equation for u- and v-component.
Gaßmann (COSMO Newsletter No. 4)

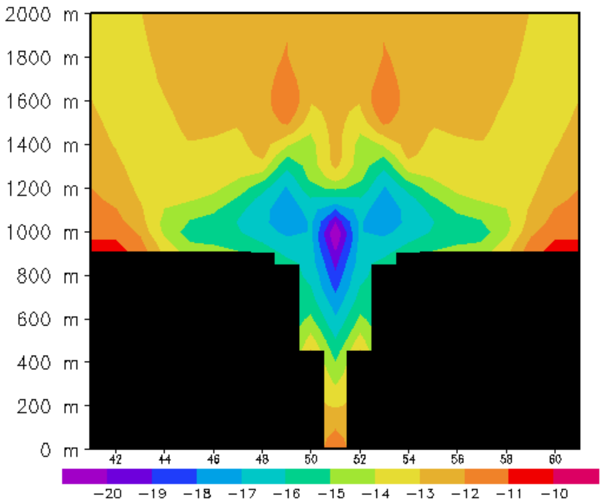
cold pool – problem in narrow valleys

is essentially induced by pressure gradient term

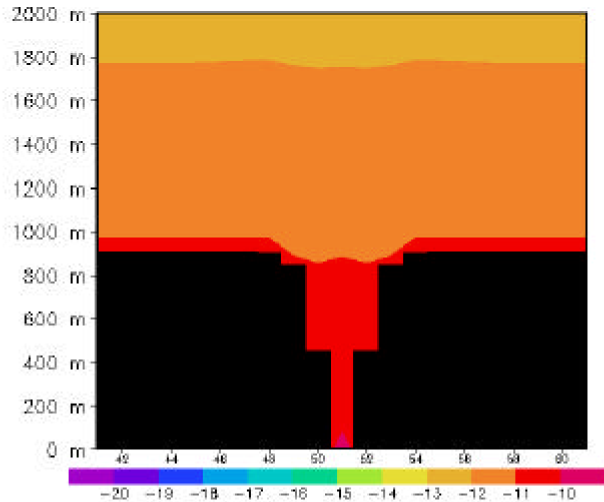


T (°C)

starting point



after 1 h



after 1 h

modified version:

pressure gradient on z-levels, if

|metric term| > |terrain follow. te

J. Förstner, T. Reinhardt

Numerical questions currently considered:

- Properties of time-splitting (Runge-Kutta) schemes:
  - convergence (in linear/nonlinear cases, dynamical core alone/with physics, ...)
  - accuracy
  - conservation
- Implicit vertical advection for dynamic variables
- fast-waves-solver (strong conservation form, ...)
- upper radiative boundary condition
- Physics-Dynamics coupling
  - originally changed due to an underestimation of precipitation in convective events because of too weak coupling
  - problems in turbulence scheme (?)
  - Bryan, Fritsch (2002) test case
- (terrain-foolowing coordinate in steep terrain)
  Cold pools in narrow, deep valleys

ENDE

Time-splitted Euler forward step