A mass, thermodynamic and lateral boundary inconsistency in the LM equations - the possible origin of a pressure bias ?

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Contents

- Original budget equations
- How to derive the prognostic equations of temperatur and pressure
- Reconsideration of approximations on the way to the LM equations
- Qualitative discussion and possible consequences to reformulate the $T-$ and $p'-$ equation

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In memory of Günther Doms
and his marvellous LM documentation !

- Original budget equations for mean flow -

- momentum equations
- total mass equation (continuity equation)
- water constituent equations
- enthalpy equation

elimination of total density

equation of state

$$\overline{\rho}\,\hat{c}_p \frac{\hat{d}\,\hat{T}}{d\,t} = \frac{\hat{d}\,\overline{p}}{d\,t} + \overline{Q}_h$$

continuity equation hidden behind !

$$\frac{\hat{d}\,\overline{p}}{d\,t} = -\left(\hat{c}_p/\hat{c}_v\right)\overline{p}\,\nabla\cdot\hat{\mathbf{v}} + \left(\hat{c}_p/\hat{c}_v - 1\right)\overline{Q}_h + \left(\hat{c}_p/\hat{c}_v\right)\overline{Q}_m$$

- Prognostic equations of temperature and pressure -

Common approximations :

- Turbulence averaging symbols dropped
- In heat and moisture source terms molecular fluxes neglected
- Latent heat of vapourisation and sublimation assumed constant
- Specific heats of moist air replaced by specific heat of dry air

$$\rho c_{pd} \frac{dT}{dt} = \frac{dp}{dt} + \underline{Q_h}$$

$$\frac{dp}{dt} = -\left(c_{pd}/c_{vd}\right) p \nabla \cdot \mathbf{v} + \left(c_{pd}/c_{vd} - 1\right) \underline{Q_h} + \left(c_{pd}/c_{vd}\right) \underline{Q_m}$$

Q_h , Q_m → heat - and moisture sources

Definition of heat source term

- Günther's (2002) notation -

$$Q_T := \frac{1}{\rho\, c_{pd}} Q_h$$

$$Q_T = -\frac{1}{\rho\, c_{pd}} \nabla \cdot \left(\mathbf{H} + \mathbf{R} \right) + \left(\frac{L_v}{c_{pd}} S^l + \frac{L_s}{c_{pd}} S^f \right)$$

Turbulent heat flux

Radiation flux

Diabatic heating due to cloud microphysical sources per unit mass

$$-\frac{1}{\rho\, c_{pd}} \nabla \cdot \mathbf{H} := -\frac{1}{\rho\, c_{pd}} \nabla \cdot c_{pd}\, \overline{\rho \mathbf{v}'' T}$$

$$\approx -\frac{1}{\rho\, c_{pd}} \nabla \cdot c_{pd}\, \bar{\pi}\, \bar{\rho}\, \overline{\mathbf{v}'\theta'} \approx \frac{1}{\rho\sqrt{G}} \frac{\partial}{\partial \zeta} \left(\frac{\pi \rho K_h}{\sqrt{G}} \frac{\partial \theta}{\partial \zeta} \right)$$

$$S^l = I^l / \rho$$

$$S^f = I^f / \rho$$

Definition of moisture source term

- Günther‘s (2002) notation -

$$Q_M := \frac{1}{\rho\, c_{pd}}\, Q_m$$

$$Q_M = \frac{R_d\, T}{\rho\, c_{pd}}\left(1-\frac{R_v}{R_d}\right)\nabla\cdot\mathbf{F}^{\mathbf{v}} - \frac{R_v\, T}{c_{pd}}\left(S^l + S^f\right) + \frac{R_d\, T}{\rho\, c_{pd}}\left[\nabla\cdot\left(\mathbf{F}^{\mathbf{l}}+\mathbf{F}^{\mathbf{f}}\right)+\nabla\cdot\left(\mathbf{P}^{\mathbf{l}}+\mathbf{P}^{\mathbf{f}}\right)\right]$$

Cloud heat sources

Turbulent flux for water constituents

$$\mathbf{F}^{\mathbf{x}} = \overline{\rho\, \mathbf{v}''\, q^x} \approx \overline{\rho\, w''\, q^x}\,\mathbf{k} \approx \frac{\rho\, K_h}{\sqrt{G}}\frac{\partial\, q^x}{\partial\, \zeta}\,\mathbf{k}$$

$$x = v, l, f$$

Precipitation (gravitational diffusion) fluxes

$$\mathbf{P}^{l,f} = -q^{l,f}\left|\mathbf{v}_T^{l,f}\right|\mathbf{k}$$

Reconsidering the derivation of prognostic temperature- and pressure equations in the LM

$$\frac{dT}{dt} = \frac{1}{\rho c_{pd}} \frac{dp}{dt} + Q_T$$

$$\frac{dp}{dt} = -\left(c_{pd}/c_{vd}\right) p \nabla \cdot \mathbf{v} + \left(c_{pd}/c_{vd} - 1\right) \rho c_{pd} Q_T + \left(c_{pd}/c_{vd}\right) \rho c_{pd} Q_M$$

1. In pressure equation heat- and moisture source terms neglected
2. dp/dt in T - equation eliminated after neglecting these terms

$$\frac{dT}{dt} = -\frac{p}{\rho c_{vd}} \nabla \cdot \mathbf{v} + Q_T$$

$$\frac{dp}{dt} = -\left(c_{pd}/c_{vd}\right) p \nabla \cdot \mathbf{v}$$

3. Formal addition of moist convection tendency, computational mixing, lateral and upper boundary relaxation terms

Final prognostic LM - equations for T and p'

$$\frac{dT}{dt} = -\frac{p}{\rho c_{vd}} \nabla \cdot \mathbf{v} + Q_T + M_T^{MC} + M_T^{CM} - \mu_b \left(T - T_b \right) - \mu_R(z) \left(T - T_b \right)$$

$$\frac{dp}{dt} = -\left(c_{pd} / c_{vd} \right) p \nabla \cdot \mathbf{v} + M_{p'}^{CM} - \mu_b \left(p' - p'_b \right) - \mu_R(z) \left(p' - p'_b \right)$$

The way to come to this result is wrong and leads to insufficient equations !

We meet with a wrong combination of p'- and T - equation, each used at a different level of approximation, which is physically inconsistent.

This operation is equivalent to the application of a wrong continuity equation producing a mass deficiency

Heating - and moisture source terms are not properly incorporated in the T - equation

The direct influence of heating and moisture terms in the pressure tendency equation is ignored

The lateral and vertical boundary relaxation technique in the T- equation is incomplete.

Consistent derivation of T- and p'- equation

1. Heating- and moisture source terms are retained in the pressure equation
2. Addition of moist convection tendency, computational mixing, lateral and upper boundary relaxation terms in both equations

$$\frac{dT}{dt} = \frac{1}{\rho c_{pd}} \frac{dp}{dt} + Q_T + M_T^{MC} + M_T^{CM} - \mu_b \left(T' - T_b' \right) - \mu_R(z) \left(T' - T_b' \right)$$

$$\frac{dp}{dt} = -\left(c_{pd} / c_{vd} \right) p \nabla \cdot \mathbf{v} + \left(c_{pd} / c_{vd} - 1 \right) \rho c_{pd} \left(Q_T + M_T^{MC} \right) +$$

$$+ \left(c_{pd} / c_{vd} \right) \rho c_{pd} Q_M + M_{p'}^{CM} - \mu_b \left(p' - p_b' \right) - \mu_R(z) \left(p' - p_b' \right)$$

3. Only now eliminating individual pressure tendency dp/dt in the first law of thermodynamics !

$$\frac{dT}{dt} = -\frac{p}{\rho c_{vd}} \nabla \cdot \mathbf{v} + $$

$$+ \left(c_{pd}/c_{vd}\right)\left(Q_T + Q_M + M_T^{MC}\right) + \left(M_T^{CM} + \frac{1}{\rho c_{pd}} M_{p'}^{CM}\right) -$$

$$-\mu_b\left[\left(T' - T_b'\right) + \frac{1}{\rho c_{pd}}\left(p' - p_b'\right)\right] - \mu_R(z)\left[\left(T' - T_b'\right) + \frac{1}{\rho c_{pd}}\left(p' - p_b'\right)\right]$$

$$\frac{dp}{dt} = -\left(c_{pd}/c_{vd}\right) p \nabla \cdot \mathbf{v} + \left(c_{pd}/c_{vd} - 1\right)\rho c_{pd}\left(Q_T + M_T^{MC}\right) +$$

$$+ \left(c_{pd}/c_{vd}\right)\rho c_{pd} Q_M + M_{p'}^{CM} - \mu_b\left(p' - p_b'\right) - \mu_R(z)\left(p' - p_b'\right)$$

new !

$$\frac{dT}{dt} = -\frac{p}{\rho c_{vd}} \nabla \cdot \mathbf{v} + Q_T + M_T^{MC} + M_T^{CM} - \mu_b\left(T - T_b\right) - \mu_R(z)\left(T - T_b\right)$$

$$\frac{dp}{dt} = -\left(c_{pd}/c_{vd}\right) p \nabla \cdot \mathbf{v} + M_{p'}^{CM} - \mu_b\left(p' - p_b'\right) - \mu_R(z)\left(p' - p_b'\right)$$

old !

- Lateral Davies - boundary relaxation technique -

Günther's notation (2002)

For temperature a different lateral boundary treatment compared to all other variables :

$$\frac{T^{n+1} - T^{*}}{\Delta t} = -\mu_b \left(T^{n+1} - T_b^{n+1} \right) - \frac{\mu_b}{c_{pd}\,\rho^{n}} \left(p'^{\,n+1} - p_b'^{\,n+1} \right)$$

$$T^{n+1} = T^{*} - \alpha \left(T^{*} - T_b^{n+1} \right) - \alpha \left(1 - \alpha \right) \left(\frac{p'^{\,*} - p_b'^{\,n+1}}{c_{pd}\,\rho^{n}} \right)$$

$$\alpha = \frac{\Delta t\, \mu_b}{1 + \Delta t\, \mu_b} \quad \Rightarrow \quad 1 - \tanh\left(\frac{d}{2\,\Delta s} \right) \quad ; \quad 0 < \alpha \le 1$$

Maximal influence of pressure boundary data on temperature within boundary zone at $\alpha = 0.5$ **.** **- Rayleigh damping similar.**

- Conclusions -

- The prognostic pressure - and temperature equations have been reconsidered and reformulated.
- It is shown that the well-known heat- and moisture source/sink terms are necessary to be taken into account in the pressure equation in order to allow in small-scale simulations the direct influence of thermal and moisture effects on the pressure field - and so on the wind field too.
- Ignoring these terms or having some inconsistent approximation of them in the pressure equation is equivalent to a hidden mass budget error in the model.
- Furthermore, even the heating terms in the final prognostic temperature equation appear to be not correct, if for the elimination of the individual pressure tendency a pressure equation is applied where the diabatic terms are already ignored or incomplete.
- It is important to note that it is necessary not to apply selective approximations in one equation, which after this approximation is used to combine with another equation having less and/or different (inconsistent) approximations.
- On this ground we are going to pursue a new project in favour of physically improving the given LM equations beyond of any improvement of parameterisation.