Hierarchical Bayes Ensemble Kalman Filter

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Methodological problems in the EnKF we intend to alleviate in the new filter

1. EnKF analysis equations are derived from the assumption that the background-error covariance matrix B is exact. But this cannot be the case in high dimensions.

2. EnKF uses ad-hoc regularization of the ensemble covariance matrix B: spatial localization plus covariance inflation. Both are *not* theoretically optimal.

3. In the EnKF's analysis equations, there is no intrinsic feedback from observations to background-error statistics. This requires external adaptation or manual tuning.

Background

Myrseth and Omre (2010) proposed to remove the assumption that the background-error covariance matrix $\mathbf{B}$ and the background-error mean field $\mathbf{m}$ are known deterministic quantities, replacing it by the assumption that these are uncertain and *random*.

Bocquet (2011) assumed that $\mathbf{B}$ is random and imposed a non-informative prior for it—in order to change the (Gaussian) prior distribution of the state $\mathbf{x}$ to a more realistic continuous mixture of Gaussians.

Hierarchical Bayes Ensemble Filter (HBEF): principle

We follow Myrseth and Omre (2010) and:

1. Split $\mathbf{B} = \mathbf{P} + \mathbf{Q}$
2. Accordingly split the background ensemble
3. Allow observations to influence the covariances

Hierarchical Bayes Ensemble Filter (HBEF): design

In a nutshell: introduce a full-fledged secondary filter.

1. Analysis
   Update the extended control vector $\mathbf{x}, \mathbf{P}, \mathbf{Q}$ using both ensemble and observational data.
2. Forecast
   Propagate in time not only $\mathbf{x}$ (as in EnKF), but also current point estimates of $\mathbf{P}, \mathbf{Q}$ (using persistence).

Analysis: basics

From the *prior* distribution $p^f(\mathbf{x}, \mathbf{P}, \mathbf{Q})$—to the *posterior* distribution:

$$p^a(\mathbf{x}, \mathbf{P}, \mathbf{Q}) = p(\mathbf{x}, \mathbf{P}, \mathbf{Q} | \mathbf{X}^{me}, \mathbf{X}^{pe}, \mathbf{y}) \propto p^f(\mathbf{x}, \mathbf{P}, \mathbf{Q}) \cdot p(\mathbf{X}^{me}, \mathbf{X}^{pe}, \mathbf{y} | \mathbf{x}, \mathbf{P}, \mathbf{Q})$$

where

$$p(\mathbf{X}^{me}, \mathbf{X}^{pe}, \mathbf{y} | \mathbf{x}, \mathbf{P}, \mathbf{Q}) \propto p(\mathbf{X}^{me} | \mathbf{Q}) \cdot p(\mathbf{X}^{pe} | \mathbf{P}) \cdot p(\mathbf{y} | \mathbf{x})$$

Analysis: prior distributions

- The cornerstone of the HBEF is the conditional Gaussian assumption for the state:

$$\mathbf{x}|\mathbf{P},\mathbf{Q} \sim N(\mathbf{m},\mathbf{P}+\mathbf{Q})$$

- The technical assumption on the covariance matrices is the *Inverse Wishart* distribution.

Ensemble likelihoods

Assumptions:

- Members of the model-error ensemble are Gaussian given $\mathbf{Q}$
- Members of the predictability ensemble are Gaussian given $\mathbf{P}$

$$p(\mathbf{X}^{\mathbf{me}}|\mathbf{Q}) \propto |\mathbf{Q}|^{-\frac{N}{2}} \mathrm{e}^{-\frac{1}{2}\sum_{\mathrm{k}=1}^{\mathrm{N}}(\mathbf{x}_{\mathbf{k}}^{\mathbf{me}}-\mathbf{m})^{\top}\mathbf{Q}^{-\mathbf{1}}(\mathbf{x}_{\mathbf{k}}^{\mathbf{me}}-\mathbf{m})} \equiv |\mathbf{Q}|^{-\frac{N}{2}}\, \mathrm{e}^{-\frac{\mathrm{N}}{2}\,\mathrm{tr}(\mathbf{S}^{\mathrm{me}}\mathbf{Q}^{-1})}, \quad (1)$$

1. Having the model-error ensemble likelihood implies that *ensemble members can be treated as generalized observations*!
2. No need and no room for approximations, no free parameters in the ensemble likelihood!

Analysis: Computational scheme

Three options:

(i) full posterior

(ii) approximate posterior, and

(iii) approximate posterior with no feedback from observations to covariances,

The simplest option (iii):

$$\mathbf{Q}^a := \frac{\chi \mathbf{Q}^f + N\mathbf{S}^{me}}{\chi + N} \qquad \text{and} \qquad \mathbf{P}^a := \frac{\phi \mathbf{P}^f + N\mathbf{S}^{pe}}{\phi + N}. \tag{2}$$

Then, compute $\mathbf{B}^a = \mathbf{P}^a + \mathbf{Q}^a$ and use $\mathbf{B}^a$ to update the state (to compute the analysis $\mathbf{x}^a$).

Numerical experiments: A 1-D model of "truth"

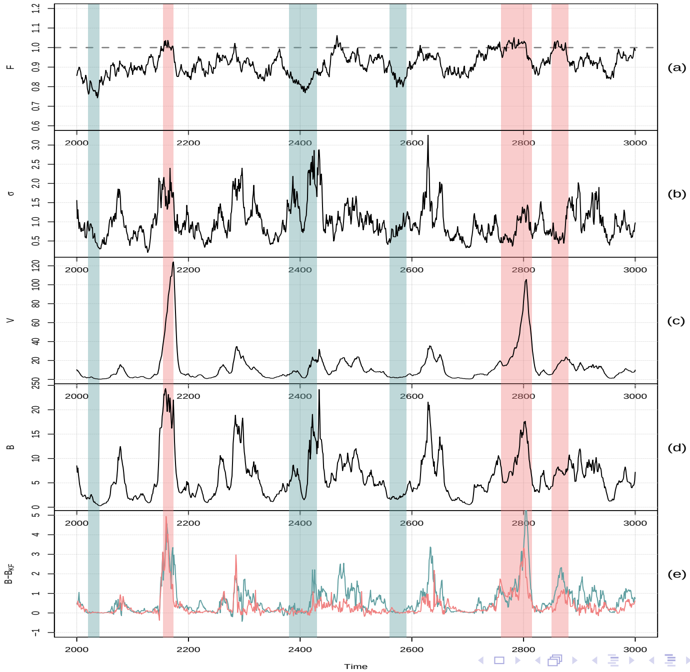
The "true" state satisfies

$$x_k = F_k x_{k-1} + \sigma_k \varepsilon_k, \tag{3}$$

where both the forecast operator F_k and the model-error standard deviation σ_k are also stochastic:

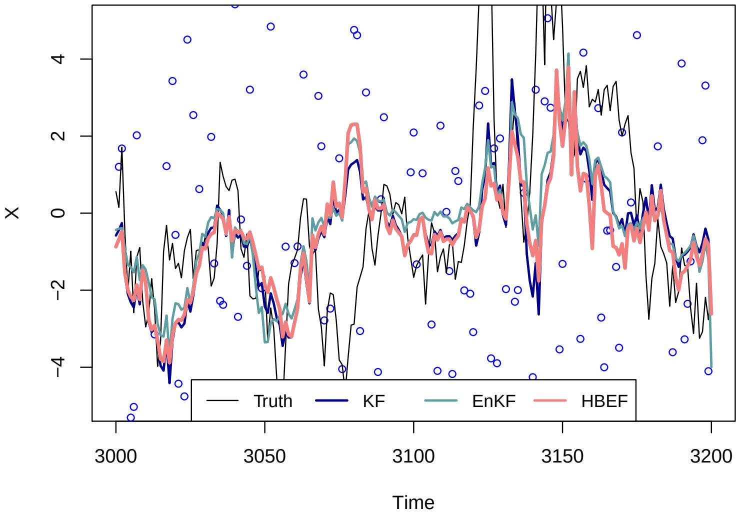
$$F_k - \bar{F} = \mu(F_{k-1} - \bar{F}) + \sigma_F \varepsilon_k^F, \tag{4}$$

and similarly for σ_k.

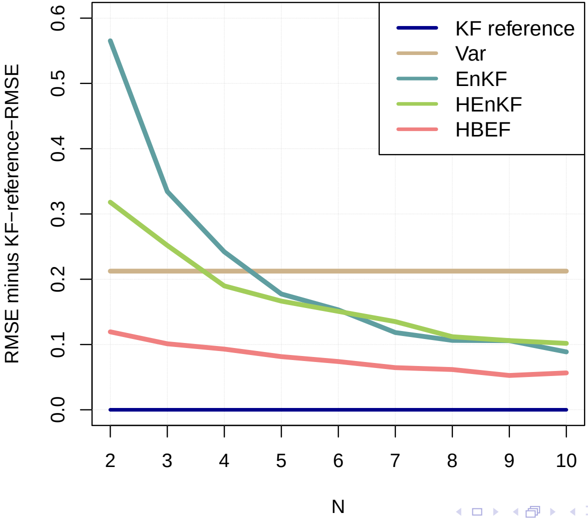


F
σ
V
B
B−B_KF
(a)
(b)
(c)
(d)
(e)
Time
2000
2200
2400
2600
2800
3000

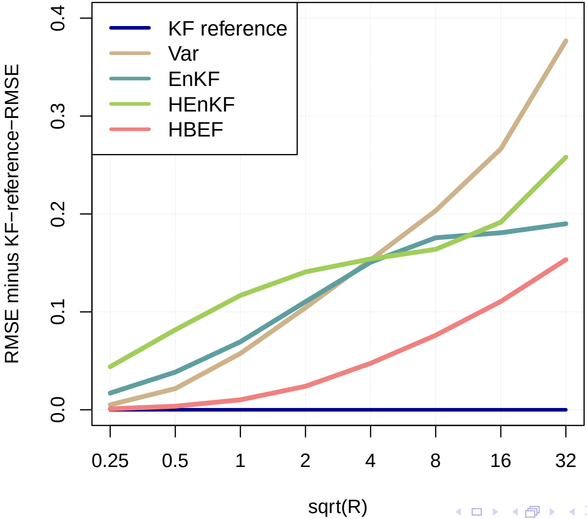
"Truth", observations (circles), and the three analyses



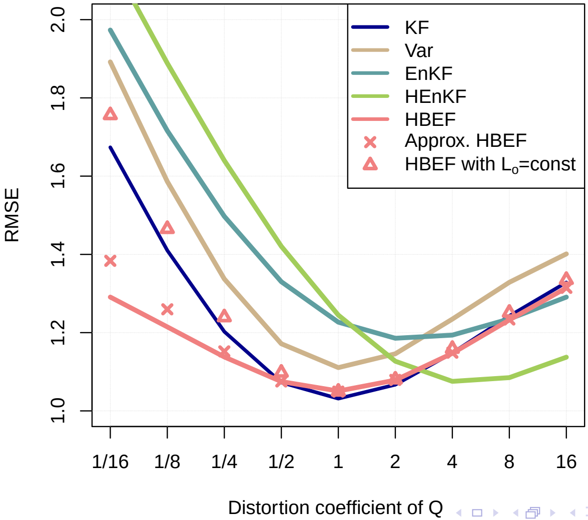
Analysis RMSEs as functions of ensemble size



Analysis RMSEs as functions of observation-error st.dev.



Analysis RMSEs when the model-error variance is distorted



Accuracy of B (performance of the secondary filter)

Filter	Error bias $\overline{B_k^*} - \overline{B_k}$	Error RMS $\mathrm{rms}\,(B_k^* - B_k)$	Mean "true" B $\overline{B_k}$
EnKF	-0.8	6.8	7.9
HBEF	-0.6	3.9	7.4
Reference KF	-0.0	0.9	7.0

Conclusions

- The new filter (HBEF) consistently addresses the inevitable uncertainty in the B matrix—by treating it as a random partially known random matrix.
- The HBEF has a full-fledged *secondary filter* that optimally updates uncertainty in the state estimates.
- Ensemble members are assimilated in the HBEF as generalized observations.
- The HBEF replaces the Gaussianity assumption by the *conditional Gaussianity*, which allows the filter to cope with non-Gaussian prior distributions of the state.
- The HBEF provides an optimized feedback from observations to background-error covariances.
- The HBEF is tested for a one-dimensional system and found significantly superior to the Var, EnKF, and HEnKF under most regimes of the system and most data assimilation setups.
- Even the cheapest version of the HBEF outperforms the traditional EnKF.

Comments

- The manuscript "Hierarchical Bayes Ensemble Kalman Filtering" is now under review in Physica D.
- The manuscript can be downloaded from
  `arXiv.org`
  or
  `ResearchGate.net`

Thank you!