Deutscher Wetterdienst
Wetter und Klima aus einer Hand

An analytic solution for linear gravity waves in a channel as a test case for solvers of the non-hydrostatic, compressible Euler equations

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For development of dynamical cores (or numerical methods in general) idealized test cases are an important evaluation tool

Idealized standard test cases for non-hydrostatic models, for which (approximated) analytic solutions exist

- stationary flow over mountains
  linear: *Queney (1947, ...), Smith (1979, ...), Baldauf (2008)*
  non-linear: *Long (1955)* for Boussinesq-approx. atmosphere
- non-stationary, linear expansion of gravity waves in a channel
  Skamarock, Klemp (1994) for Boussinesq-approx. atmosphere

most of the other idealized tests only possess 'known solutions' gained by other numerical models.

There exist even less analytic solutions which use the exact equations, i.e. in a sense that a numerical model would converge to this solution.
One exception is given here:
linear expansion of gravity/sound waves in a channel

Non-hydrostatic compressible, 2D, Euler equations in a flat channel (shallow atmosphere) on an f-plane

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f v,$$

$$\frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v = -f u,$$

$$\frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right),$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = c_s'^2 \left(\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \right),$$

$$T = \frac{p}{R\rho},$$

$$c_s' = \sqrt{\frac{c_p}{c_v} RT},$$

most LAMs using the compressible equations should be able to exactly use these equations in the dynamical core

For analytic solution only one further approximation is needed: <u>linearisation</u> (= *controlled* approximation) around an **isothermal, steady, hydrostatic** atmosphere at rest (f≠0 possible) or with a constant basic flow U_0 (and f=0)

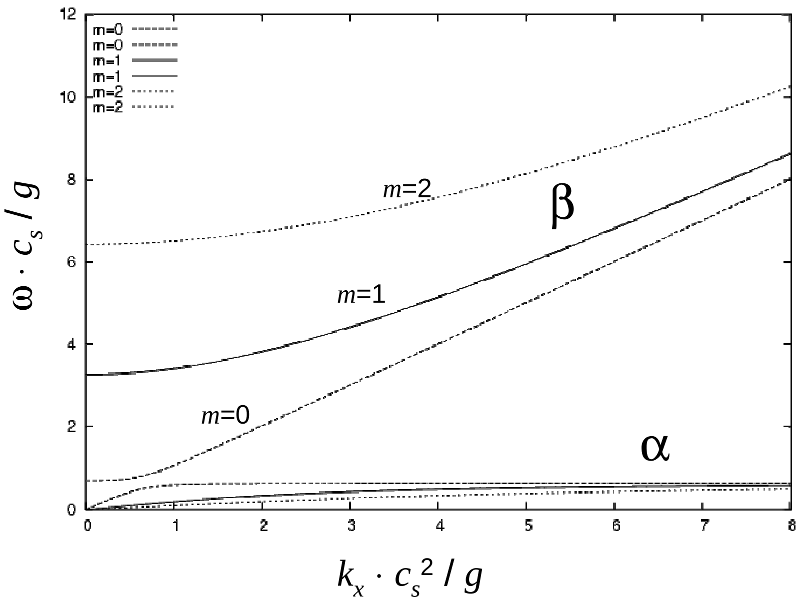
Bretherton-, Fourier- and Laplace-Transformation →

Analytic solution for the Fourier transformed vertical velocity *w*

$$\hat{w}_b(k_x, k_z, t) = -\frac{1}{\beta^2 - \alpha^2}\left[-\alpha \sin \alpha t + \beta \sin \beta t + (f^2 + c_s^2 k_x^2)\left(\frac{1}{\alpha}\sin \alpha t - \frac{1}{\beta}\sin \beta t\right)\right] g \frac{\hat{\rho}_b(k_x, k_z, t=0)}{\rho_s}$$

analogous expressions for $u_b(k_x, k_z, t)$, ...

The frequencies α, β are the gravity wave and acoustic branch, respecticely, of the dispersion relation for compressible waves in a channel with height H;
kz = (π / H) · m

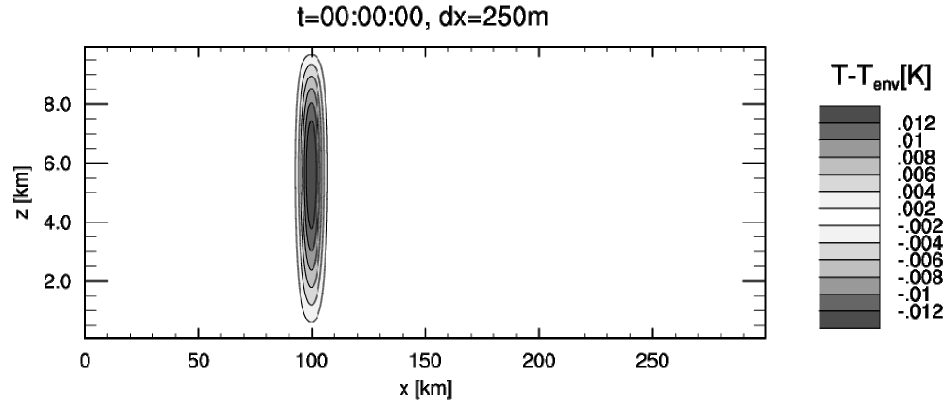


Test case: expansion of gravity and sound waves by the initialisation of a weak warm bubble:

$$T'(x,z,t=0) = \Delta T \cdot e^{\frac{1}{2}\delta z} \cdot e^{-\frac{(x-x_c)^2}{d^2}} \cdot \sin \pi \frac{z}{H}$$

$$p'(x,z,t=0) = 0$$

weak bubble ΔT =0.01 K → linear regime



Small scale test

with a basic flow U_0=20 m/s

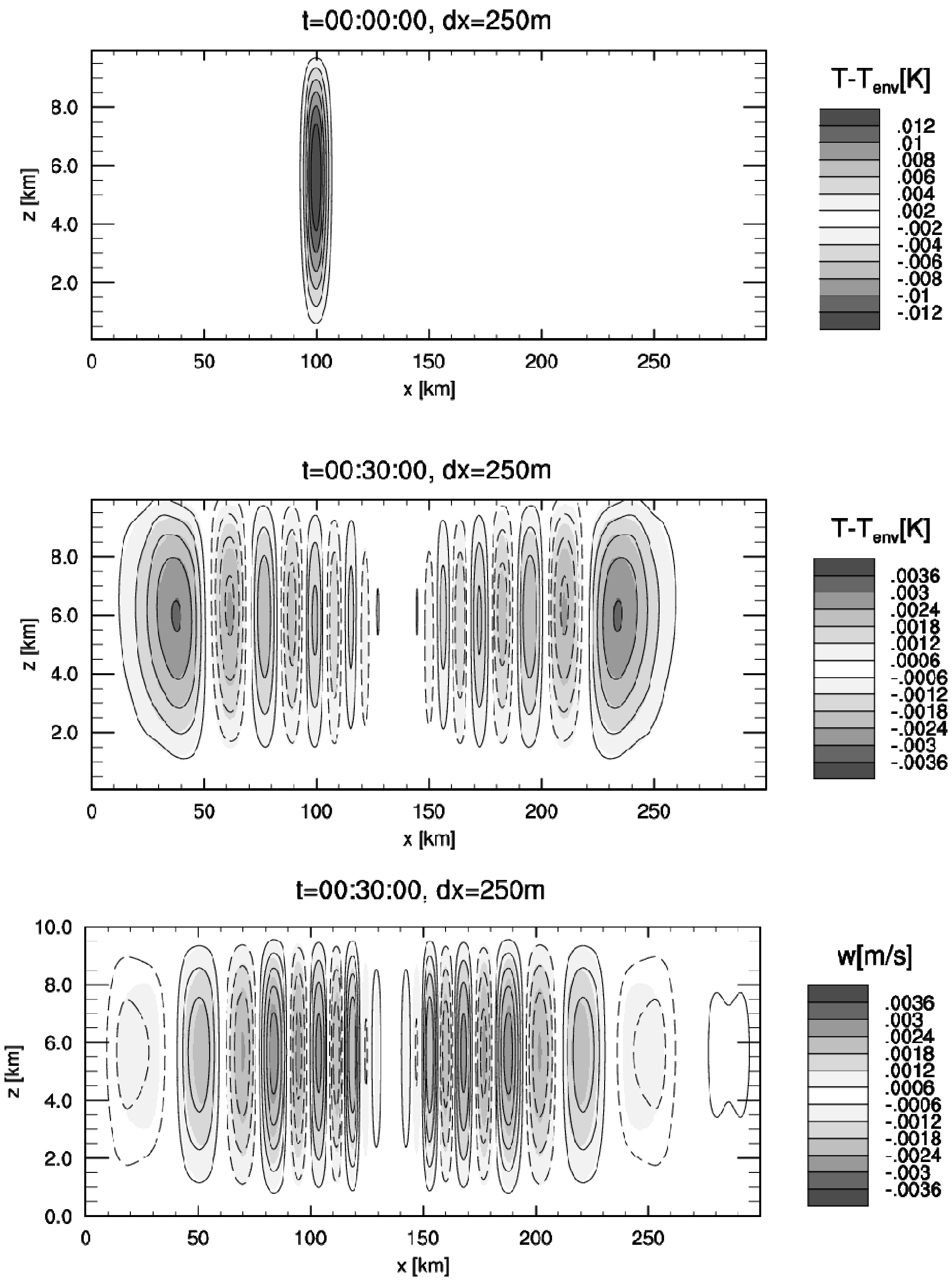
f=0

Initialisation similar to

Skamarock, Klemp (1994)

Lines: analytic solution

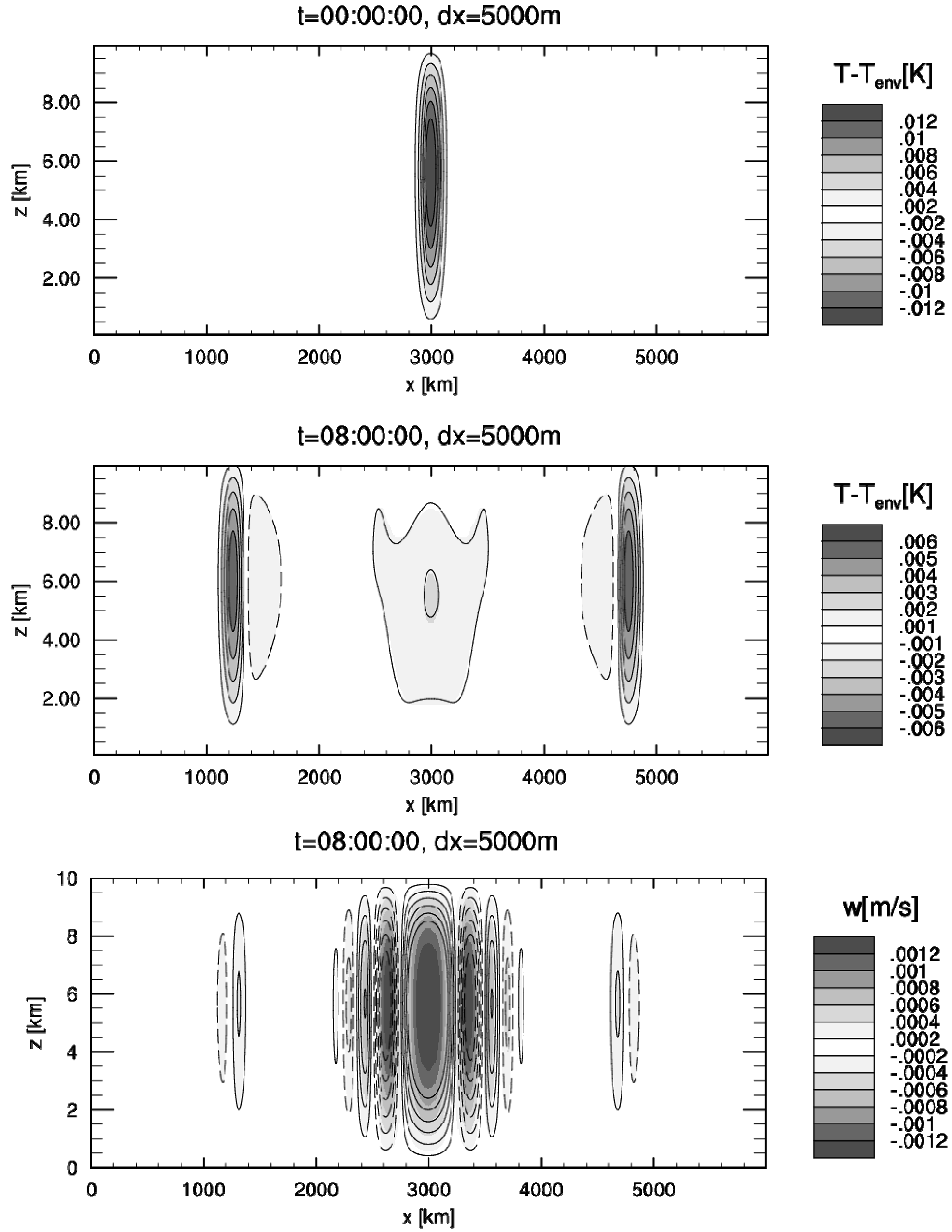
Shaded: COSMO



Large scale test
U_0=0 m/s
f=0.0001 1/s

Initialisation similar to
Skamarock, Klemp (1994)

Lines: analytic solution
Shaded: COSMO



Convergence properties of COSMO

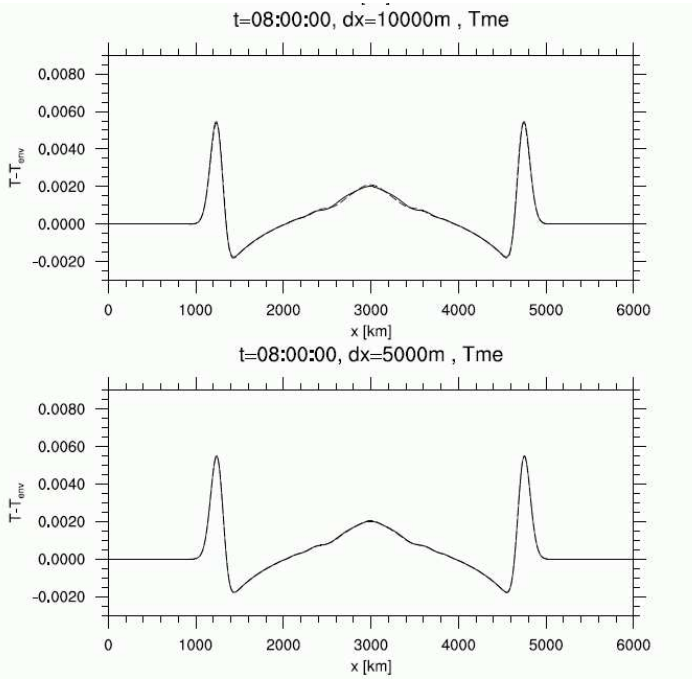


Figure 6. w' (top) and T' (bottom) for $\Delta x = 10000$ m and $\Delta x = 5000$ m; large-scale test case with $f > 0$. Comparison between the analytic solution (solid line) and COSMO with reduced time step $\Delta t = 0.5$ s and no off-centring for vertically Crank-Nicholson sound waves treatment (dashed).

Table 2. Small scale test case: convergence of w' for COSMO.

Δx [m]	Δt [s]	L_∞-error	L_2-error
1000	20.0	$1.53e{-}3$	$3.94e{-}4$
500	10.0	$1.08e{-}3$	$3.59e{-}4$
250	5.0	$8.22e{-}4$	$2.93e{-}4$
125	2.5	$5.53e{-}4$	$1.99e{-}4$
50	1.0	$2.73e{-}4$	$9.74e{-}5$

Table 7. Large scale test case: convergence of w' for COSMO with reduced time step and no off-centring for sound.

Δx [km]	Δt [s]	L_∞-error	L_2-error
20.0	0.5	$2.38e{-}3$	$4.62e{-}4$
10.0	0.5	$2.26e{-}3$	$5.15e{-}4$
5.0	0.5	$6.30e{-}4$	$1.46e{-}4$
2.5	0.5	$2.91e{-}4$	$6.41e{-}5$

A new dynamical core based on Discontinuous Galerkin methods

Project 'Adaptive numerics for multi-scale flow', DFG priority program 'Metström'

currently cooperation between DWD, Univ. Freiburg, Univ. Warwick

Simulations with DUNE
developed at Universities of Freiburg, Heidelberg, Stuttgart
(Dedner et al., 2007)

Discontinuous Galerkin Method of 2nd order and
2nd order time integration (Heun-scheme)

Simulations performed by *Slavko Brdar* (Univ. Freiburg, Germany)

Convergence properties of DUNE

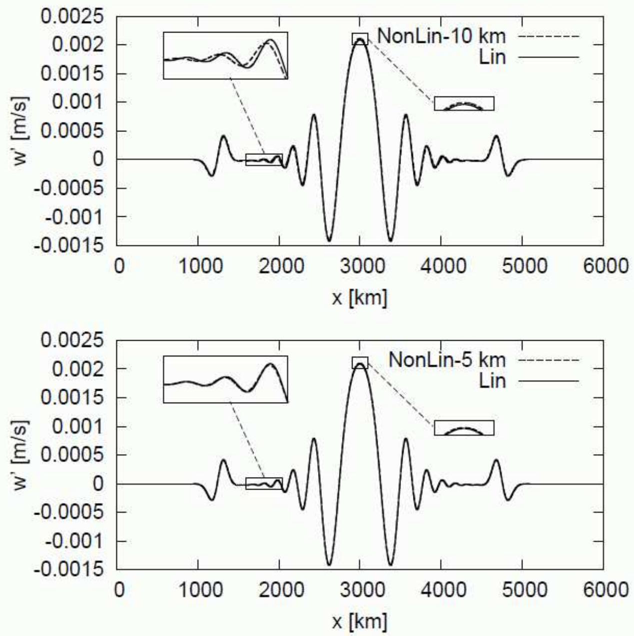


Figure 4. Large scale test case with $f > 0$: The vertical velocity w' for *(up)* $\Delta x = 10$ km and *(below)* $\Delta x = 5$ km with DUNE in comparison to the analytic solution (solid line) at $t = 28800$ s and $\Delta z = \Delta x/160$.

Table 4. Small scale test case: convergence of w' for DUNE.

Δx [m]	Δt [s]	L_∞-error	L_2-error
1000.0	0.48	1.15e–3	1.99e–4
500.0	0.24	2.03e–4	3.24e–5
250.0	0.12	3.36e–5	5.65e–6
125.0	0.06	6.56e–6	1.30e–6
62.5	0.03	3.26e–6	8.66e–7

Table 9. Large scale test case: convergence of w' for DUNE.

Δx [km]	Δt [s]	L_∞-error	L_2-error
20.0	1.3e–1	2.92e–4	7.14e–5
10.0	6.5e–2	7.52e–5	1.95e–5
5.0	3.3e–2	2.57e–5	6.04e–6
2.5	1.6e–2	1.79e–5	3.64e–6

Summary

- An analytic solution of the compressible, non-hydrostatic Euler equations was derived
→ a reliable solution for a well known test exist and can be used not only for qualitative comparisons but even as a reference solution for convergence tests

- 'standard' approximations used: f-plane, shallow atmosphere, can be easily realised in a every atmospheric model

- only one further approximation: linearisation

- For fine enough resolutions COSMO has a spatial-temporal convergence rate of about 0.8. The spatial convergence rate is ~1.

M. Baldauf, S. Brdar: An Analytic solution for Linear Gravity Waves in a Channel as a Test for Numerical Models using the Non-hydrostatic, Compressible Euler Equations, submitted to *Quart. J. Roy. Met. Soc.*