Localization for Ensemble Kalman Filters -Basics and Recent Results

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Tikhonov Regularization uset 3dVar Kalman Filter - Update B Ensem die Kalman Filter (EnKF) and Local Ensemblewian Silter (LenKF) eisemblewiar Silvar, EnKF and LETKF



Example 1: Original Dynamics and Measurement Points



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Example 2: Original Dynamics and Measurement Points



Tikhonov Regularization uset 3dVar Kalman Filter - Update B Ensem die Kalman Filter (EnKF) and Local Ensemble: Kalman Filter (LEnKF) Semble: Tor Java, EnKF and LETKF



Example 3: Original Dynamics and Measurement Points



Tikhonov Regularization and 30Var Katomi Filter Jugiate B Ensemble Kalman Filter (EnKF) and Local Ensemble Kalman Filter (LEnKF) Complexity of Stark Enst LETKF





- Setup a simple example system with a travelling "quadratic" front
- Dynamics is a shift operator to the right Dynamics is called
 M : φ_k → φ_{k+1}
- Measurement points indicated by white "+", either widely distributed or with some measurement free area

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- Vectors φ or φ_k at time t_k in X = ℝ^N describe the field values in a regular grid.
- 5) Measurement Operator *H* selects the measurements, next neightbor interpolation.
- 6) Measurement vector f, time dependence: f_k at time t_k

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Tikhonov Regularization and 3dVar Kalman Filter. Up ate B Ensen de Kalman Filter (EnKF) and Local Ensemble Kalman Filter (LEnKF)

Outline

Tikhonov Regularization and 3dVar

Kalman Filter: Update B

Ensemble Kalman Filter (EnKF) and Local Ensemble Kalman Filter (LEnKF)

Examples for 3dVar, EnKF and LETKF



Basic Approach

Let *H* be the data operator mapping the state φ onto the measurements *f*. Then we need to find φ by solving the equation

$$H\varphi = f \tag{1}$$

When we have some initial guess $arphi^{(b)},$ we transform the equation into

$$H(\varphi - \varphi^{(b)}) = f - H\varphi^{(b)}$$
⁽²⁾

with the incremental form

$$\varphi = \varphi^{(b)} + H^{-1}(f - H\varphi^{(b)}). \tag{3}$$



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Regularization 1

Consider an equation

$$H\varphi = f \tag{4}$$

where H^{-1} is unstable or unbounded.

$$H\varphi = f$$

$$\Rightarrow H^*H\varphi = H^*f$$

$$\Rightarrow (\alpha I + H^*H)\varphi = H^*f.$$
(5)

where $(\alpha I + H^*H)$ has a stable inverse!

Tikhonov Regularization: Replace H^{-1} by the stable version

$$R_{\alpha} := \left(\alpha I + H^* H\right)^{-1} H^* \tag{6}$$

with regularization parameter $\alpha > 0$.

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Regularization 2: Least Squares

Tikhonov regularization is equivalent to the minimization of

$$J(\varphi) := \left(\alpha \|\varphi\|^2 + \|H\varphi - f\|^2 \right)$$
(7)

The normal equations are obtained from first order optimality conditions

$$\nabla_{\varphi}J \stackrel{!}{=} 0. \tag{8}$$

Differentiation leads to

$$0 = 2\alpha\varphi + 2H^*(H\varphi - f)$$

$$\Rightarrow \quad 0 = (\alpha I + H^*H)\varphi - H^*f, \tag{9}$$

which is our well-known Tikhonov equation

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Covariances and Weighted Norms

Usually, the relation between variables at different points is incorporated by using **covariances** / **weighted norms**:

$$J(x) := \left(\|\varphi - \varphi^{(b)}\|_{B^{-1}}^2 + \|H\varphi - f\|_{B^{-1}}^2 \right)$$
(10)

The update formula is now

$$\varphi = \varphi^{(b)} + (B^{-1} + H^* R^{-1} H)^{-1} H^* R^{-1} (f - H \varphi^{(b)})$$
(11)

or

$$\varphi = \varphi^{(b)} + BH^*(R + HBH^*)^{-1}(f - H\varphi^{(b)}).$$
 (12)

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How to Update B in KF

Kalman Update Formula for the covariance matrix *B* (with *R* error covariance matrix)

$$(B_{k+1}^{(a)})^{-1} = (B_{k+1}^{(b)})^{-1} + H^* R^{-1} H, \quad k = 1, 2, \dots$$
(13)

and for the mean

$$\varphi_{k+1}^{(a)} = \varphi_k^{(b)} + B_{k+1}^{(b)} H^* (HB_{k+1}^{(b)} H^* + R)^{-1} (f_{k+1} - H\varphi_k^{(b)})$$
(14)

for k = 1, 2, ... with $B_{k+1}^{(b)}$ being the propagated covariance matrix from $B_k^{(a)}$.

Theorem

For linear systems and linear observation operators 4dVar and the Kalman Filter are equivalent.

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EnKF: B via Ensemble of states



Use stochastic estimator

$$B = \mathbb{E}\left\{ (\varphi - \overline{\varphi})(\varphi - \overline{\varphi})^T \right\}$$
(15)

Tikhonov Regularization and 3dVar Kalman Filter, Uprate B Ensemble Kalman Filter (EnKF) and Local Ensemble Kalman Filter (LEnKF)

EnKF Analysis 1

Updates are

$$\varphi_k = \varphi_k^{(b)} + BH^*(R + HBH^*)^{-1}(f_k - H\varphi_k^{(b)})$$

• The stochastic estimator is given by

$$Q_{k} = \left(\varphi^{(1)} - \overline{\varphi}, ..., \varphi^{(L)} - \overline{\varphi}\right), \qquad B = \frac{1}{L-1}Q_{k}Q_{k}^{T} \qquad (16)$$

• We solve the 3dVar update in the low-dimensional subspace

$$X_{k}^{(L)} := \operatorname{span}\left\{\varphi^{(1)} - \overline{\varphi}, ..., \varphi^{(L)} - \overline{\varphi}\right\}$$
(17)

by

$$\varphi_k = \varphi_k^{(b)} + Q_k \underbrace{Q_k^* H^* (R + HQ_k Q_k^* H^*)^{-1}}_{(f_k - H\varphi_k^{(b)})} (f_k - H\varphi_k^{(b)})$$

regularized projection

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Tikhonov Regularization and 3dVar Kalmon Filter Usate B Ensemble Kalman Filter (EnKF) and Local Ensemble Kalman Filter (LEnKF



Error Estimate for EnKF



The minimizer of the data term only

$$u_{k,0}^{(a)} = P_k(arphi_{true,k} - arphi_k^{(b)})$$

is given by an orthogonal projection.

The minimizer $u_k^{\left(a
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 $J(u) = \|u\|_{B^{-1}}^2 + \|(f_k - H\varphi_k^{(b)}) - Hu\|_{B^{-1}}^2$

is on the line between $u_{k,0}^{(a)}$ and u=0 in $X_k^{(L)}.$

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Tikhonov Regularization of 3dVar Kalmon Filter Under B Ensemble Kalman Filter (EnKF) and Local Ensemble Kalman Filter (LEnKF)



Error estimates for EnKF

Theorem

Assume that H is injective, that we study true data f_k and consider the EnKF with data term only. Then, the error $E_{k,0}$ in the k-th step of the EnKF is given by

$$E_{k,0} = d_{H^*R^{-1}H}(X_k^{(L)}, \varphi_{true,k} - \varphi^{(b)}).$$

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Assume that H is injective and that we study true data f_k . Then, the error E_k in the k-th step of the EnKF is estimated by

$$\|\varphi_{\textit{true},k} - \varphi^{(b)}\|_{H^*R^{-1}H} \geq E_k \geq d_{H^*R^{-1}H}(X_k^{(L)}, \varphi_{\textit{true},k} - \varphi^{(b)})$$

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LEnKF Basic Idea: Localization

• We employ localization:

$$arphi_k = arphi_k^{(b)} + B_{loc}H^*(R + HB_{loc}H^*)^{-1}(f_k - Harphi_k^{(b)})$$

• Here, the localization is given by

$$B_{loc} = C \circ B \tag{18}$$

(where \circ denotes the Schur product, i.e. pointwise matrix multiplication) with

$$C_{j,k} := e^{-|p_i - p_k|^2/\sigma}, \ j, k = 1, ..., N.$$

There are different alternative ways how to carry out localization!!

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Examples for 3dVar, EnKF and LETKF

Example 1: 3dVar





Example 1: Original



Example 1: EnKF





Example 1: LEnKF





Example 1: Original



nKF) 'KF



Example 2: 3dVar





Example 2: Original



Example 2: EnKF





Example 2: LEnKF





Example 2: Original



Example 3: 3dVar





Example 3: Original



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Example 3: EnKF





Example 3: LEnKF





Example 3: Original





- 1. What type of localization is optimal for our systems?
- 2. Adaptive localization depending on the data density?
- 3. Dynamics and update rates, how to choose them?
- 4. Convection resolving scale and localization strategies?
- 5. Basic conceptional and convergence questions are unresolved!
- 6. How to estimate and include model error
- 7. Localization and nonlocal remote sensing data?



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Current work points / scientific and operational questions

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Ensemble Control ...

Control the ensemble to optimally shape $X_k^{(L)}$ locally ...!





Many Thanks!